Fundamental groups and Diophantine geometry

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Motives created by Grothendieck in the 60's.

$$X \in \mathsf{Var}_F \mapsto [X] \in \mathsf{Mot}_F$$

supposed to be a universal cohomology theory.

$[X]$ is an *abelianization* of X, in that Mot is an abelian category.

What (the theory of) motives cannot do: provide information about non-abelian Diophantine sets.

X hyperbolic curve over F

J_X: Jacobian of X.

There should be an isomorphism:

$$J_X \simeq \mathsf{Ext}(\mathbb{Z}, H_1(X))$$

where $H_1(X)$ is the motivic H_1 of X with integral coefficients.

With respect to Hodge structures:

$$J_X(\mathbb{C}) \simeq H_1(X,\mathbb{Z})\backslash H_1(X,\mathbb{C})/F^0$$

where the latter object classifies exact sequences

$$0\to H_1(X)\to E\to \mathbb{Z}\to 0$$

of integral Hodge structures.

In étale cohomology: Kummer theory and the conjecture of Birch and Swinnerton-Dyer.

Application to Diophantine sets supposed to arise from choice of a basepoint $b \in X(F)$. This determines an embedding

$$X \hookrightarrow J_X$$

via

$$x \mapsto [0 \to H_1(X) \to H_1(X \setminus \{b, x\}) \to \mathbb{Z} \to 0]$$

and hence, an inclusion

$$X(F) \hookrightarrow J_X(F)$$

Algebraic construction due to Weil where J_X is interpreted as an abelian variety. Application to the finiteness of $X(F)$ in mind. (Mordell conjecture, i.e. Faltings' theorem.)

Advantage: $J_X(F)$ is an abelian group.

Disadvantage: $J_X(F)$ is an abelian group.

$J_X(F)$ can be infinite in general, exactly because you can add points to get more points.

Problem is the intrinsically abelian nature of the category of motives. So, even in the best of possible worlds (i.e., where all conjectures are theorems), the category of motives does not (by itself) touch on some very basic Diophantine phenomena.

Weil's fantasy (1938, 'Generalization of abelian functions'):

Importance of developing 'non-abelian mathematics.'

Ingredients should involve moduli of vector bundles and fundamental groups.

Weil thought such theories should have application to arithmetic. Plausible that he had the Mordell conjecture in mind.

Weil's paper began the theory of vector bundles on curves, leading eventually to Narashimhan-Seshadri, Donaldson, Simpson, etc.

No arithmetic theory of π_1 at the time.

60's: Grothendieck's theory of π_1.

X: smooth variety over F.

$Cov(X)$: category of finite étale coverings of X.

$b : \mathrm{Spec}(K) \to X$: a geometric point of X.

$f_b : Cov(X) \to$finite sets,

$$\begin{array}{ccc} Y & & Y_b \\ \downarrow & \mapsto & \downarrow \\ X & & b \end{array}$$

Pro-finite fundamental group:

$$\widehat{\pi}_1(X, b) := \mathsf{Aut}(f_b)$$

Difficult definition to use. But spectacular application in resolution of Weil conjectures (as opposed to Weil's fantasy).

However, direction of application is *transverse* to Weil's fantasy.

Given variety B and base point $b \in B$

$\mathbb{Q}_l$-lisse sheaves on $B \leftrightarrow$ continuous $\mathbb{Q}_l$-reps of $\pi_1(B, b)$.

Thus, to study the arithmetic of a variety Y, put it into a family

$$\begin{array}{ccc} Y & \hookrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ b & \hookrightarrow & B \end{array}$$

and study the action of $\pi_1(B, b)$ on $H^i(Y, \mathbb{Q}_l)$.

Simply an analogue of Gal$(\bar{F}/F)$ acting on $H_1(\bar{X}, \mathbb{Q}_l)$. (However, note that the π_1 can act naturally on the geometric fiber over a closed point.)

That is to say, no role for the *vertical* fundamental group $\pi_1(Y)$.

New proposal by Grothendieck in the 80's:

Anabelian geometry.

Important role for pro-finite torsor of paths:

$$\widehat{\pi}_1(X; a, b) := \mathrm{Isom}(f_a, f_b)$$

Flexible variation of points highlights the power of Grothendieck's definition.

$\bar{X} := X \otimes_F \bar{F}$.

$a, b \in X(F)$: rational points, regarded as geometric points.

$\Gamma := \mathrm{Gal}(\bar{F}/F)$

Then Γ acts on $Cov(\bar{X})$ preserving the fiber functors f_a, f_b, so

$$\hat{\pi}_1(\bar{X}; a, b)$$

has a Γ-action.

Natural maps

$$\widehat{\pi}_1(\bar{X}; b, c) \times \widehat{\pi}_1(\bar{X}; a, b) \rightarrow \widehat{\pi}_1(\bar{X}; a, c)$$

compatible with Γ-action.

Thus, $\widehat{\pi}_1(\bar{X}; b, c)$ becomes a $\Gamma-$equivariant right-torsor for $\widehat{\pi}_1(\bar{X}, b)$.

Classified by

$$H^1(\Gamma, \widehat{\pi}_1(\bar{X}, b))$$

Get a map

$$\kappa : X(F) \to H^1(\Gamma, \widehat{\pi}_1(\bar{X}, b))$$

$$x \mapsto [\widehat{\pi}_1(\bar{X}; b, x)]$$

Note, when composed with

$$\widehat{\pi}_1(\bar{X}, b) \to \widehat{\pi}_1(\bar{X}, b)^{ab} \simeq \widehat{T} J_X$$

coincides with map

$$\kappa^{ab} : X(F) \to H^1(\Gamma, \widehat{T} J_X)$$

coming from Kummer theory.

However, $H^1(\Gamma, \widehat{T} J_X)$ is abelian, and contains all the points of J_X (even if we impose all reasonable local conditions).

Grothendieck's section conjecture:

When X is a compact hyperbolic curve,

$$\kappa : X(F) \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

is a bijection.

Grothendieck and Deligne expected:

Section conjecture $\Rightarrow$ Faltings' theorem

Initial reasoning appears to have been erroneous.

κ is very much studied already in *abelian* contexts.

(X, b) an elliptic curve. Then $\hat{\pi}_1(\bar{X}, b))$ is abelian, and

$$X(F) \to \widehat{X(F)} \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

comes from the usual connecting homomorphism for the Kummer exact sequence.

In this case, the image lies inside a subspace

$$H^1_f(\Gamma, \hat{\pi}_1(\bar{X}, b)) \subset H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

satisfying local Selmer conditions.

The bijectivity of

$$\widehat{X(F)} \rightarrow H^1_f(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

is an important part of the Birch and Swinnerton-Dyer conjecture.

That is, *the section conjecture is a natural non-abelian generalization of the finiteness of Sha*.

For varieties over finite fields, κ^{ab} coincides with the reciprocity map of geometric class field theory restricted to cycles of the form $(x)-(b)$. Thus, reciprocity law is a kind of 'abelianized' section conjecture over finite fields.

Section conjecture and relation to Mordell appears quite hard. But spectacular progress on other aspects of anabelian geometry by Pop, Nakamura, Tamagawa, Mochizuki,...

60's Grothendieck: Theory of Motives.

80's Grothendieck: Anabelian Geometry.

Intermediate theory: Theory of the motivic fundamental group (Deligne).

Can redo some of Grothendieck's ideas in this context. In particular, find a good substitute for

Section conjecture $\Rightarrow$ Mordell conjecture

That is, marginal progress on application of π_1 to Diophantine finiteness.

NB: we have nothing to say about section conjecture.

Basic idea: map

$$x \in X(F) \mapsto [\pi_1(\bar{X}; b, x)]$$

taking values in a classifying space for torsors exists for any suitable theory of π_1.

Technical part: p-adic analysis as Galois theory (e.g. p-adic Hodge theory).

Recall Chabauty's method. Assume

$$rk J_X(F) < \text{dim} J_X \qquad (*)_2$$

Then $X(F)$ is finite.

Proof:

$$\begin{array}{ccccc}
X(F) & \hookrightarrow & X(F_v) & \to & T_e J_X(F_v) \\
\downarrow & & \downarrow & log \nearrow & \downarrow \alpha \\
J_X(F) & \hookrightarrow & J_X(F_v) & & F_v
\end{array}$$

α: linear function on the g-dimensional $\mathbb{Q}_p$-vector space $T_e J_X$ such that $\alpha \circ log$ vanishes on $J_X(F)$.

Re-interpret Chabauty using p-adic Hodge theory and ideas of Bloch-Kato-Kolyvagin.

$X/\mathbb{Q}$ genus 1.

Kato produces $c \in H^1(\Gamma, H_1(\bar{X}, \mathbb{Q}_p))$ such that the map

$$H^1(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)(1)) \rightarrow$$

$$\rightarrow H^1(\Gamma_p, H^1(\bar{X}, \mathbb{Q}_p)(1)) \stackrel{exp*}{\rightarrow} F^0 H_1^{DR}(X_p)$$

takes

$$c \mapsto L_X(1)\alpha$$

α a global 1-form. Using it to annihilate points

$$x \in X(\mathbb{Q}) \subset X(\mathbb{Q}_p) \subset T_e X = H_1^{DR}/F^0$$

gives finiteness of $X(\mathbb{Q})$ if $L(1) \neq 0$.

Chabauty's diagram can also be replaced by

$$\begin{array}{ccccc} X(F) & \hookrightarrow & X(F_v) & \rightarrow & T_eJ_X(F_v) \\ \downarrow & & \downarrow & log \nearrow & \downarrow \\ H^1_f(\Gamma, H_1(\bar{X})) & \hookrightarrow & H^1_f(\Gamma_v, H_1(\bar{X}) & & F_v \end{array}$$

Finiteness follows whenever

$$Im(H^1_f(\Gamma, H_1(\bar{X}, \mathbb{Q}_p)))$$

is not Zariski dense.

By combining this diagram with Tate duality, we see that Chabauty's method is an *imprecise* higher genus analogue of Kolyvagin-Kato. That is, the hypothesis on rank implies the *existence* of an

$$\alpha \in H^1(\Gamma, H_1(\bar{X}, \mathbb{Q}_p))$$

whose component at v is non-zero under the dual exponential map.

But an extension of the method unique to higher genus arises from promoting the above to a whole sequence of diagrams:

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook] \arrow[d] & X(\mathbb{Q}_p) \arrow[d] \arrow[dr] & \\
H^1_f(\Gamma, U_n^{et}) \arrow[r] & H^1_f(G_p, U_n^{et}) \arrow[r, "D"] & U_n^{DR}/F^0 \arrow[d, "\alpha"] \\
& & \mathbb{Q}_p
\end{tikzcd}$$

U's are different components of the motivic fundamental group of X.

One component, the De Rham fundamental group of $X_{\mathbb{Q}_p}$, uses the category

$$\mathrm{Un}(X_{\mathbb{Q}_p})$$

of unipotent vector bundles with flat connection.

That is, the objects are $(\mathcal{V}, \nabla)$, vector bundles $\mathcal{V}$ on $X_{\mathbb{Q}_p}$ equipped with flat connections

$$\nabla : \mathcal{V} \rightarrow \Omega_{X/S} \otimes \mathcal{V}$$

that admit a filtration

$$\mathcal{V} = \mathcal{V}_n \supset \mathcal{V}_{n-1} \supset \cdots \supset \mathcal{V}_1 \supset \mathcal{V}_0 = 0$$

by sub-bundles stabilized by the connection, such that

$$(\mathcal{V}_{i+1}/\mathcal{V}_i, \nabla) \simeq (\mathcal{O}^r_{X_{\mathbb{Q}_p}}, d)$$

Associated to $b \in X$ get

$$e_b : \mathsf{Un}(X_{\mathbb{Q}_p}) \to \mathsf{Vect}_{\mathbb{Q}_p}$$

The *De Rham fundamental group*

$$U^{DR} := \pi_{1,DR}(X_{\mathbb{Q}_p}, b)$$

is the pro-unipotent pro-algebraic group that represents

$$\mathsf{Aut}^{\otimes}(e_b)$$

(Tannaka dual) and the path space

$$P^{DR}(x) := \pi_{1,DR}(X; b, x)$$

represents

$$\mathsf{Isom}^{\otimes}(e_b, e_x)$$

The pro-unipotent p-adic étale fundamental group

$$U^{et}$$

and étale path spaces

$$P^{et}(x)$$

defined in the same way using the category of unipotent $\mathbb{Q}_p$ local systems.

$Z^i \subset U$ defined by descending central series.

$$U_i = Z^i \backslash U$$

Can push out torsors as well to get

$$P_i = Z^i \backslash P \times U$$

Extra structures:

U^{et}, P^{et}: Γ-action.

U^{DR}, P^{DR}: Hodge filtrations and Frobenius-actions.

$H^1_f(\Gamma, U_n)$ Selmer varieties classifying torsors that satisfying natural local conditions. Most important one: Restriction to G_p trivializes over B_{cr}.

U^{DR}/F^0 classifies U^{DR}-torsors with Frobenius action and Hodge filtration. Map

$$X(\mathbb{Q}) \rightarrow H^1_f(\Gamma, U_n)$$

associates to a point the torsor $P_n^{et}(x)$. Similarly

$$X(\mathbb{Q}_p) \rightarrow U_n^{DR}/F^0$$

uses torsor $P_n^{DR}(x)$. Compatibility provided by non-abelian p-adic comparison isomorphism.

Return to diagram.

$$\begin{array}{ccccc}
X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) & & \\
\downarrow & & \downarrow & \searrow & \\
H^1_f(\Gamma, U_n^{et}) & \to & H^1_f(G_p, U_n^{et}) & \stackrel{D}{\to} & U_n^{DR}/F^0 \\
& & & & \downarrow\alpha \\
& & & & \mathbb{Q}_p
\end{array}$$

Assume $(*)_n$:

$$Im(H^1_f(\Gamma, U_n^{et}) \subset U_n^{DR}/F^0$$

not Zariski dense.

$(*)_n$ implies finiteness of integral or rational points.

Note that all ingredients predicted by Weil have gone into the construction of this diagram.

α algebraic function that vanishes on global points. Can be expressed in terms of p-adic iterated integrals, e.g., p-adic multiple polylogarithms in the case of $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Note: Special values of such functions related to L-values. However, α here not precise enough to have such specific relations as in abelian case.

$(*)_n$ for $n >> 0$ implied by various motivic conjectures.

-Bloch-Kato conjecture on image of p-adic Chern class map.

-Fontaine-Mazur conjecture on geometric Galois representations.

-X affine. Jannsen's conjecture on vanishing of

$$H^2_f(\Gamma, H^n(\bar{V}, \mathbb{Q}_p(r))$$

for large r.

All provide bounds on dimensions of

$$H^1_f(\Gamma, U_n)$$

Precise form: All classes in

$$H^1_f(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1))$$

are motivic. That is,

Motives $\rightarrow H^1_f(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1))$

surjective.

Analogous to

$$X(F) \rightarrow H^1(\Gamma, \hat{\pi}(\bar{X}, b))$$

surjective.

Sort of substitute for

'Section conjecture $\Rightarrow$ Mordell conjecture.'

Can prove $(*)_n$, $n >> 0$ for:

-genus one hyperbolic curves.

-CM elliptic curves of rank 1 (minus the origin).

-Other hyperbolic curves subject to 'locality' of certain restricted ramification Galois groups.

To improve the situation, much more precise study of

$$H^1_f(\Gamma, U_n^{et}) \rightarrow H^1_f(G_p, U_n^{et}) \xrightarrow{D} U_n^{DR}/F^0$$

related to p-adic L-functions desirable, with the aim of arriving at a *precise non-abelian analogue* of the Kolyvagin-Kato method.

For example, start with pairing

$$\mathrm{Ext}_{\Gamma_p}(U, \mathbb{Q}_p(1)) \times H^1(\Gamma_p, U) \rightarrow H^2(\Gamma_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

and try to produce good global elements in $\mathrm{Ext}_{\Gamma_p}(U, \mathbb{Q}_p(1))$.