Motives and Diophantine geometry V

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Several conjectures:

(BSD) $E/\mathbb{Q}$ elliptic curve. Then

$$\delta : \widehat{E(\mathbb{Q})} \to H^1_f(\Gamma, \hat{T}E)$$

is surjective.

(Bloch-Kato) $X/\mathbb{Q}$ smooth projective variety. Then

$$ch_{n,r} : K^{(r)}_{2r-n-1}(X) \otimes \mathbb{Q}_p \to H^1_g(\mathbb{Q}, H^n(\bar{X}, \mathbb{Q}_p(r)))$$

is surjective.

(Fontaine-Mazur) $X/\mathbb{Q}$ smooth projective variety. Then

$$\text{Mixed Motives} \to H^1_g(\Gamma, H^n(\bar{X}, \mathbb{Q}_p(r))$$

is surjective.

(Grothendieck) $X/\mathbb{Q}$ compact hyperbolic curve, $b \in X(\mathbb{Q})$. Then

$$X(\mathbb{Q}) \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

is surjective.

Grothendieck and Deligne expected

Grothendieck's conjecture $\Rightarrow$ Mordell conjecture.

In fact,

B-K or F-M $\Rightarrow$ Mordell conjecture over $\mathbb{Q}$.

Lesson: Importance of abelian techniques even in the study of non-abelian objects.

$X/\mathbb{Q}$: compact hyperbolic curve.

Un$^B(X(\mathbb{C}))$: category of unipotent $\mathbb{Q}$-local systems on $X(\mathbb{C})$.

Un$^{et,p}(\bar{X})$: category of unipotent $\mathbb{Q}_p$-local systems on $\bar{X}^{et}$.

Un$^{DR}(X_{\mathbb{Q}_p})$: category of unipotent vector bundles with flat connections on $X_{\mathbb{Q}_p}$.

b: a point of $X(\mathbb{Q})$

x: point in either $X(\mathbb{C})$, $X(\mathbb{Q})$, or $X(\mathbb{Q}_p)$ depending on context.

Fiber functors $f_x^B, f_x^{et}, f_x^{DR}$ taking values in Vect$_{\mathbb{Q}}$, Vect$_{\mathbb{Q}_p}$.

So far defined

$U^B = U_{\mathbb{Q}}(\pi_1(X(\mathbb{C}), b))$ and

$P^B(x) = \mathsf{Isom}^{\otimes}(f_b^B, f_x^B)$ for $x \in X(\mathbb{C})$.

$U^{et} = U_{\mathbb{Q}_p}(\widehat{\pi}_1(\bar{X}, b))$ and

$P^{et}(x) = \mathsf{Isom}^{\otimes}(f_b^{et}, f_x^{et})$ for $x \in X(\mathbb{Q})$

$U^{DR} = \mathsf{Aut}^{\otimes}(f_b)$ and

$P^{DR}(x) = \mathsf{Isom}^{\otimes}(f_b, f_x)$ for $x \in X(\mathbb{Q}_p)$.

Related by comparison isomorphisms:

$$P^{et}(x) \simeq P^{B}(x) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

For any embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$,

$$P^{DR}(x) \otimes \mathbb{C} \simeq P^{B}(x) \otimes \mathbb{C}$$

For p a prime of good reduction for X,

$$P^{et}(x) \otimes B_{cr} \simeq P^{DR}(x) \otimes B_{cr}$$

Also implicitly a De Rham-to-crystalline comparison isomorphism:

$$P^{DR}(x) \simeq P^{cr}(x)$$

endowing P^{DR} with a Frobenius endomorphism.

Remark on étale-to De Rham comparison: If $\mathcal{P}^{DR}(x)$ and $\mathcal{P}^{et}(x)$ are the coordinate rings of $P^{DR}(x)$ and $P^{et}(x)$, then

$$(\mathcal{P}^{et} \otimes B_{cr})^{\Gamma_p} \simeq \mathcal{P}^{DR}$$

Left hand side is the value at $P^{et}(x)$ of a non-abelian Dieudonné functor that sends U^{et} torsors with Γ_p-action to U^{DR}-torsors with Hodge filtration and Frobenius.

We will refer to the collection of these unipotent fundamental groups and path torsors as the *motivic* fundamental group and the motivic torsor of paths.

Important for us are the étale torsors with local and global Galois actions classified by

$$H^1_f(\Gamma, U^{et})$$

and

$$H^1_f(\Gamma_p, U^{et})$$

as well as the De Rham torsors classified by

$$U^{DR}/F^0$$

When we pass to quotients modulo the descending central series, we have corresponding objects

$$U^{et}_n, P^{et}_n(x), U^{DR}_n, P^{DR}_n(x)$$

and classifying spaces

$$H^1_f(\Gamma, U^{et}_n), H^1_f(\Gamma_p, U^{et}_n), U^{DR}_n/F^0$$

We also described maps

$$X(\mathbb{Q}) \rightarrow H^1_f(\Gamma, U_n^{et})$$

$$x \mapsto P^{et}(x) \qquad \text{(with } \Gamma\text{−action)}$$

$$X(\mathbb{Q}_p) \rightarrow H^1_f(\Gamma_p, U_n^{et})$$

$$x \mapsto P^{et}(x) \qquad \text{(with } \Gamma_p\text{-action)}$$

and

$$X(\mathbb{Q}_p) \mapsto U_n^{DR}/F^0$$

$$x \mapsto P^{DR}(x)$$

Can be thought of as

$$x \mapsto P^M(x),$$

the motivic torsor of paths.

The corresponding map over $\mathbb{C}$ takes values in a complex manifold of the form

$$L_n \backslash U^{DR}_{n,\mathbb{C}} / F^0$$

called the *higher Albanese manifolds*.

Usual Albanese map corresponds to $n = 2$.

Note that the map over $\mathbb{Q}_p$ has no L (periods). Same phenomenon as the global definability of log map.

$$x \mapsto P^M(x)$$

might be called the *motivic unipotent Albanese map*.

Inductive structure:

$$0 \rightarrow Z^{n+1} \backslash Z^n \rightarrow U^M_{n+1} \rightarrow U^M_n \rightarrow 0$$

Can use this to compute various dimensions.

For example, if $d_n := \dim Z^{n+1} \backslash Z^n$, then have recursive formula

$$\Sigma_{k|n} k d_k = (g + \sqrt{g^2 - 1})^n + (g - \sqrt{g^2 - 1})^n$$

and hence,

$$d_n \approx (g + \sqrt{g^2 - 1})^n / n$$

But can also analyze Galois cohomology in the étale realization.

$$H^1_f(\Gamma, U_n^{et}) \subset H^1(\Gamma_T, U_n^{et})$$

where T is the set of primes of bad reduction and p, and G_T is the Galois group of the maximal extension of $\mathbb{Q}$ with ramification restricted to T.

We have the sequence:

$$0 \to H^1(\Gamma_T, Z^{n+1}\backslash Z^n) \to H^1(\Gamma_T, U_{n+1}^{et}) \to$$

$$\to H^1(\Gamma_T, U_n^{et}) \xrightarrow{\delta} H^2(\Gamma_T, Z^{n+1}\backslash Z^n)$$

which is exact in the sense that

$$H^1(\Gamma_T, U_{n+1}^{et})$$

is a torsor for the vector group

$$H^1(\Gamma_T, Z^{n+1}\backslash Z^n)$$

over the kernel of δ.

Note: All cohomology sets naturally have the structure of algebraic varieties.

Can use this sequence to estimate the growth in dimension of $H^1(\Gamma_T, U_n^{et})$ and hence, of $H^1_f(\Gamma_T, U_n^{et})$.

Recall diagram:

$$\begin{array}{ccccc}
X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) & & \\
\downarrow & & \downarrow & \searrow & \\
H^1_f(\Gamma, U_n^{et}) & \rightarrow & H^1_f(\Gamma_p, U_n^{et}) & \stackrel{D}{\rightarrow} & U_n^{DR}/F^0 \\
& & & & \downarrow\alpha \\
& & & & \mathbb{Q}_p
\end{array}$$

If we show

$$\mathrm{dim} H^1_f(\Gamma, U_n^{et}) < U_n^{DR}/F^0 \qquad (*)_n$$

for some n, then get finiteness of $X(\mathbb{Q})$.

Depends also on local description of algebraic functions on U_n^{DR}/F^0 pulled back to $X(\mathbb{Q}_p)$ as p-adic iterated integrals.

Reduces to Chabauty's method when $n = 2$.

Can prove this kind of statement for hyperbolic curves of genus 0 and genus 1 CM of rank 1.

Other curves depending on conditions on G_T.

Controlling the dimension uses the Euler characteristic formula

$$\mathrm{dim} H^1(\Gamma_T, Z^{n+1}\backslash Z^n) - \mathrm{dim} H^2(\Gamma_T, Z^{n+1}\backslash Z^n)$$

$$= \mathrm{dim}(Z^{n+1}\backslash Z^n)^-$$

where the negative superscript refers to the (-1) eigenspace of complex conjugation. By comparison with complex Hodge theory, we see that the right hand side is

$$d_n/2$$

for n odd.

When the genus is zero,

$$Z^{n+1}\backslash Z^n \simeq \mathbb{Q}_p(n)^{d_n}$$

and

$$H^2(\Gamma_T, Z^{n+1}\backslash Z^n) = 0$$

for $n \geq 2$ and Therefore, eventually,

$$\mathrm{dim} H^1(\Gamma_T, U_n^{et}) < \mathrm{dim} U_n^{DR}/F^0$$

The general implications over $\mathbb{Q}$ rely on bounds of the form

$$\dim H^2(\Gamma_T, Z^{n+1}\backslash Z^n) \leq P(n)g^n$$

implied by Bloch-Kato or Fontaine-Mazur.

We have a surjection

$$H^2(\Gamma_T, H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n}) \rightarrow H^2(\Gamma_T, Z^{n+1}\backslash Z^n) \rightarrow 0$$

and an exact sequence

$$0 \rightarrow Sh^2(H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n}) \rightarrow H^2(\Gamma_T, H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n}) \rightarrow$$

$$\oplus_{w \in T} H^2(G_w, H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n})$$

The local groups are bounded by a quantity of the form $P(n)g^n$ using Hodge-Tate decomposition and the monodromy-weight filtration.

Furthermore,

$$Sh^2(H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n}) \simeq (Sh^1(H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1)))^*$$

with the latter group defined by

$$0 \to Sh^1(H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1)) \to H^1(\Gamma_T, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1))$$

$$\to \oplus_{w \in T} H^1(G_w, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1)))$$

But either Bloch-Kato or Fontaine-Mazur implies that $Sh^1_n = 0$ for $n \geq 2$.

Note that we have in place all the ingredients predicted by Weil's fantasy:

-Vector bundles;

-π_1;

-application to arithmetic.

Not yet a 'π_1-proof' of finiteness. But at least marginal progress.

Remarks on future direction.

Need a non-abelian method of Kolyvagin-Kato.

$X/\mathbb{Q}$ genus 1.

Kato produces $c \in H^1(\Gamma, H_1(\bar{X}, \mathbb{Q}_p))$ such that the map

$$H^1(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)(1)) \rightarrow$$

$$\rightarrow H^1(\Gamma_p, H^1(\bar{X}, \mathbb{Q}_p)(1)) \stackrel{exp*}{\rightarrow} F^0 H_1^{DR}(X_p)$$

takes

$$c \mapsto L_X(1)\alpha$$

α a global 1-form. Using it to annihilate points (local-global duality)

$$x \in X(\mathbb{Q}) \subset X(\mathbb{Q}_p) \subset T_e X = H_1^{DR}/F^0$$

gives finiteness of $X(\mathbb{Q})$ if $L(1) \neq 0$.

Should be promoted to a precise study of map

$$H^1_f(\Gamma, U_n^{et}) \to H^1_f(G_p, U_n^{et}) \xrightarrow{D} U_n^{DR}/F^0$$

enabling the *construction* of a function vanishing on the image.

Foundational work:

Non-abelian dualities in Galois cohomology.

Difficult part:

Production of a non-abelian 'dual' global cohomology class.

For example, start with pairing

$$\mathrm{Ext}_{\Gamma_p}(U, \mathbb{Q}_p(1)) \times H^1(\Gamma_p, U) \to H^2(\Gamma_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

and try to produce good global elements in $\mathrm{Ext}_{\Gamma_p}(U, \mathbb{Q}_p(1))$.

Fantasy:

Non-abelian (p-adic) L-functions?

Should take values in a homogenous space for the fundamental group?