Motives and Diophantine geometry IV

July 6, 2006

Today: Some constructions and one example.

Theory of motives ($\mathbb{Q}$-homological) admits many techniques, but looses much information.

Anabelian geometry (pro-finite) is expected to completely encode certain schemes, but allows far fewer techniques.

$$\widehat{\pi}_1 \to \cdots \text{very far} \cdots \to H_1$$

$$\widehat{\pi}_1 \to \cdots \to \text{motivic } \pi_1 \to H_1$$

Motivic π_1 is still quite close to homology, and hence, admits homological techniques. But non-abelian!

Review of topological situation. X topological space. $\pi_1(X, b)$ is a discrete group. $H_1(X, \mathbb{Q})$ is a vector space, hence, an algebraic group. Relation between the two has to do with certain completions.

For any discrete group G (finitely generated), various ways of turning it into an algebraic group. Easiest one to describe is the unipotent completion $U(G)$ w.r.t. a field F of characteristic zero.

Start with the group ring $F[G]$, has the natural structure of a Hopf algebra:

$$\Delta : F[G] \rightarrow F[G] \otimes F[G]$$

defined by

$$\Sigma_g c_g [g] \mapsto \Sigma_g c_g g \otimes g$$

In fact, easy to show that G is recovered exactly as the group-like elements of $F[G]$:

$$\{a \in F[G] | \Delta(a) = a \otimes a\}$$

Let I be the augmentation ideal defined by the exact sequence

$$0 \rightarrow I \rightarrow F[G] \stackrel{deg}{\rightarrow} F \rightarrow 0$$

and define the completed group algebra

$$R := \varprojlim F[G]/I^n$$

Any finite-dimensional F-rational representation

$$\rho : G \rightarrow \mathrm{Aut}(V)$$

extends to

$$\rho : F[G] \rightarrow \mathrm{End}(V)$$

This extends to

$$\rho : R \rightarrow \mathrm{End}(V)$$

iff the original representation is *unipotent*.

The comultiplication Δ extends to one on R, and we define the $F-$unipotent completion $U_F(G)$ of G to be the group-like elements in R.

Its image in any of the quotients $F[G]/I^n$ is a unipotent algebraic group over F, giving $U(G)$ the structure of a pro-unipotent, pro-algebraic group.

When X is a space and $G=\pi_1(X,b)$, then there is a functorial geometric approach to the definition of $U^B(X)=U_{\mathbb{Q}}(\pi_1(X,b))$.

Consider

$$\mathsf{Un}^B(X)$$

the category of unipotent locally constant sheaves of $\mathbb{Q}$ vector spaces on X. Can consider this as a $\mathbb{Q}$-linearized version of the category of covering spaces. Associated to points, there are fiber functors

$$f_b : \mathsf{Un}^B(X) \rightarrow \mathsf{Vect}_{\mathbb{Q}}$$

Fact:

$$U^B(X) \simeq \mathsf{Aut}^{\otimes}(f_b)$$

Then this allows us to construct 'unipotent completions of path spaces' as

$$P^b(X, x) := \mathrm{Isom}(f_b, f_x)$$

Pro-algebraic variety.

When we start with the fundamental group $\pi_1(X, b)$ of a manifold X, we can construct $U^B_{\mathbb{C}}(X) = U_{\mathbb{C}}(\pi_1)$ using differential forms, i.e., Chen's iterated integrals.

Given n complex 1-forms α_i on X, get a function $[a_1|a_2|\cdots|\alpha_n]$ on paths as follows. If $\gamma : I \to X$ is a piecewise smooth paths, write $\gamma(\alpha_i) = f_i(t_i)dt_i$. Then

$$[a_1|a_2|\cdots|\alpha_n](\gamma) =$$

$$\int_0^1 f_1(t_1) \int_0^{t_1} f_2(t_2) \int_0^{t_2} \cdots \int_0^{t_{n-1}} f_n(t_n) dt_n dt_{n-1} \cdots dt$$

Subject to a certain algebraic condition, such functions are homotopy invariant, linear combinations of these naturally form a Hopf algebra $II(X)$ of iterated integrals.

Fact:

$$U_{\mathbb{C}}(\pi_1(X,b)) \simeq \mathsf{Spec}(II(X))$$

Another formulation: If A is the coordinate ring of $U^B_{\mathbb{Q}}(\pi_1)$, then $A \otimes \mathbb{C} \simeq II(X)$. Using this, can put natural $\mathbb{Q}$-Hodge structure on A. Actually need to do this carefully to encode dependence on basepoint, but will ignore this subtlety here. The Hodge filtration $F^iA \subset A$ is by ideals that are compatible with the comultiplication, and hence, define subgroups

$$F^iU^B_{\mathbb{C}} = Z(F^{1-i})$$

In fact, get a Hodge structure on all $P^B(X,x)$, which varies with the base-point x.

Will often omit X from notation and write, e.g., $P^B(x)$.

When X is a smooth variety defined over $\mathbb{Q}$, can also define $U^{DR}(X)$, the De Rham fundamental group of X, which is a pro-unipotent, pro-algebraic group over $\mathbb{Q}$ admitting a purely algebraic definition.

Uses the category

$$\mathrm{Un}(X)$$

of unipotent vector bundles with flat connection.

That is, the objects are $(\mathcal{V}, \nabla)$, vector bundles $\mathcal{V}$ on X equipped with flat connections

$$\nabla : \mathcal{V} \rightarrow \Omega_{X/S} \otimes \mathcal{V}$$

that admit a filtration

$$\mathcal{V} = \mathcal{V}_n \supset \mathcal{V}_{n-1} \supset \cdots \supset \mathcal{V}_1 \supset \mathcal{V}_0 = 0$$

by sub-bundles stabilized by the connection, such that

$$(\mathcal{V}_{i+1}/\mathcal{V}_i, \nabla) \simeq (\mathcal{O}^r_{X_{\mathbb{Q}_p}}, d)$$

Associated to $b \in X$ get

$$e_b : \mathsf{Un}(X_{\mathbb{Q}}) \rightarrow \mathsf{Vect}_{\mathbb{Q}}$$

The *De Rham fundamental group*

$$U^{DR} := \pi_{1,DR}(X_{\mathbb{Q}}, b)$$

is the pro-unipotent pro-algebraic group that represents

$$\mathsf{Aut}^{\otimes}(e_b)$$

(Tannaka dual) and the path space

$$P^{DR}(x) := \pi_{1,DR}(X; b, x)$$

represents

$$\mathsf{Isom}^{\otimes}(e_b, e_x)$$

This construction commutes with base-change, giving us an isomorphism

$$P^B(x) \otimes \mathbb{C} \simeq P^{DR}(x) \otimes \mathbb{C}$$

From this, one has a theory of non-abelian unipotent periods, including, for example, multiple zeta values: arises from taking the torsor of paths on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ going from 0 to 1 with tangential basepoints. Evaluate $\mathbb{Q}$-rational algebraic functions on P^{DR} at the point corresponding to the $\mathbb{Q}$-rational point of P^B given by the standard path.

The pro-unipotent p-adic étale fundamental group

$$U^{et}$$

and étale path spaces

$$P^{et}(x)$$

defined in the same way using the category of unipotent $\mathbb{Q}_p$ local systems.

$Z^i \subset U$ defined by descending central series.

$$U_i = Z^i \backslash U$$

Can push out torsors as well to get

$$P_i = Z^i \backslash P \times U$$

Extra structures:

U^{et}, P^{et}: Γ-action.

U^{DR}, P^{DR}: Hodge filtrations and Frobenius-actions, coming from comparison with a crystalline fundamental group and crysalline torsors of paths.

$$H^1_f(\Gamma, U_n^{et})$$

Selmer varieties classifying torsors that satisfy natural local conditions. Most important one: Restriction to G_p trivializes over B_{cr}.

U^{DR}/F^0 classifies U^{DR}-torsors with Frobenius action and Hodge filtration. Map

$$X(\mathbb{Q}) \rightarrow H^1_f(\Gamma, U_n^{et})$$

associates to a point the torsor $P_n^{et}(x)$. Similarly

$$X(\mathbb{Q}_p) \rightarrow U_n^{DR}/F^0$$

uses torsor $P_n^{DR}(x)$. Compatibility provided by non-abelian p-adic comparison isomorphism.

Regard $P^{et}(x)$ just with local Galois action of Γ_p. If $\mathcal{P}^{et}(x)$ is its coordinate ring, then

$$(\mathcal{P}^{et}(x) \otimes B_{cr})^{\Gamma_p} \simeq \mathcal{P}^{DR}(x)$$

the coordinate ring of $P^{DR}(x)$.

This construction explains the map

$$H^1_f(\Gamma, U_n) \rightarrow U_n^{DR}/F^0$$

Finish today with a brief discussion of base-point dependence, using the pro-finite case. But first, start with topological situation.

X topological space. $b, x \in X$ points. $Cov(X)$, category of covering spaces of X. f_b, f_x fiber functors defined by the points.

Fact:

$$\text{Isom}(f_b, f_x) \simeq \pi_1(X; b, x)$$

Proof: We have already discussed the map

$$\pi_1(X; b, x) \rightarrow \text{Isom}(f_b, f_x)$$

Let $p : X' \rightarrow X$ be the universal covering space of X. Choose a point $b' \in X'$ lying over $b \in X$.

Injectivity: Consider action of γ and σ on X'_b. Then lifts γ' and σ' will take b' to the same point x'. But X' is simply connected, so γ' and σ' are homotopic. So γ and σ are homotopic.

Surjectivity: Let $\phi \in \text{Isom}(f_b, f_x)$ and put $x' = \phi(b') \in X'_x$. Choose a path γ' and X' from b' to x' and let γ be its image in X.

Fact: Given any covering Y and $c \in Y_b$, there exists a unique map $f_c : X' \rightarrow Y$ such that $f(b') = c$.

So

$$\gamma(c) = \gamma f(b') = f(\gamma(b')) = f(\phi(b')) = \phi f(b') = \phi(c)$$

Actually, note that we have an isomorphism

$$\text{Isom}(f_b, f_x) \simeq X'_x (\simeq \pi_1(X; b, x))$$

Such that

$$\phi \in Isom(f_b, f_x) \mapsto \phi(b') \in X'_x$$

This fact generalizes!

E, elliptic curve over $\bar{\mathbb{Q}}$. We will describe

$$\widehat{\pi}_1(E, 0)$$

and, more generally,

$$\widehat{\pi}_1(E; 0, x)$$

Need to construct a 'universal covering space.'

Suppose $Y \to E$ is an étale cover of degree d. Then Y has genus 1 and we have a factorization

$$\begin{tikzcd} & Y \arrow[dr] & \\ d : E \arrow[ur] \arrow[rr] & & E \end{tikzcd}$$

Denote by E_d this cover $d : E \to E$.

Then the inverse system $E' = \{E_d\}$ functions as a universal covering space. Note that we have a choice of a lifting $0' = \{O_d\} \in E'_0$.

Fact:

$$\widehat{\pi}_1(E; 0, x) \simeq E'_x$$

Proof: Clearly have a map $\gamma \mapsto \gamma(0')$.

Injectivity: Let $\gamma, \sigma \in \widehat{\pi}_1(E; 0, x)$ satisfy $\gamma(0') = \sigma(0')$. Let $c \in Y_0$ for a covering Y. Then there is a map $f : E_d \rightarrow Y$ for some d. Furthermore, by translation, can make $f(0_d) = c$. An obvious argument gives us $\gamma(c) = \sigma(c)$.

Surjectivity: Left as an exercise.

This construction should make it clear that the map

$$x \mapsto [\widehat{\pi}(E; 0, x)]$$

is the same as that occurring in Kummer theory.

Similar construction for any variety X. Can construct a co-final system

$$X' = \{X_i\}$$

of finite Galois étale covers of X together with a compatible system of base-points

$$b' = \{b_i \in X_i\}.$$

In this situation,

$$\widehat{\pi}_1(X; b, x) \simeq X'_x$$

Similar in other situations: In De Rham setting, for example, can construct a universal torsor

$$(P')^{DR}$$

which is a principal U^{DR} bundle with the property that

$$(P')^{DR}_x = P^{DR}(x).$$

Next time: sketch of argument involving Diophantine finiteness.