Motives and Diophantine geometry II

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$X/\mathbb{Q}$ variety, e.g.,

$$f(x_1, x_2, \ldots, x_n) = 0, \quad f \in \mathbb{Q}[\underline{x}]$$

Given any field F, write

$$X(F)$$

for the F-points of X, that is, F-rational solutions of equation.

$$X(\mathbb{Q})$$

typical object of study in Diophantine geometry.

Naive view of Galois actions and Diophantine geometry:

$$X(\bar{\mathbb{Q}}),$$

an object of algebraic-geometric nature, has action of $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and

$$X(\mathbb{Q}) = X(\bar{\mathbb{Q}})^{\Gamma}$$

Very difficult to use this fact perhaps because of mysterious structure of Γ.

Γ-action is tractable typically for points of finite group schemes, for example,

$$A[n],$$

the n-torsion of an abelian variety A.

Can use this action to study more general points!

Review case of elliptic curve E:

$$0 \to E[n] \to E \xrightarrow{n} E \to 0$$

Taking Γ-invariants leads to

$$E(\mathbb{Q}) \xrightarrow{n} E(\mathbb{Q}) \to H^1(\Gamma, E[n])$$

and hence, an inclusion

$$E(\mathbb{Q})/n \hookrightarrow H^1(\Gamma, E[n])$$

Recall,

$$E(\mathbb{Q}) \simeq \mathbb{Z}^r \times (\text{computable finite group})$$

So $E(\mathbb{Q})/n$ for any given n determines r. Thus, can hope to use this inclusion.

Unfortunately, $H^1(\Gamma, E[n])$ infinitely generated in general, so hard to describe $E(\mathbb{Q})/n$ inside it. But can refine the description.

For a prime p, put $\Gamma_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

$$\begin{array}{ccccccc}
0\to & E(\mathbb{Q})/n & \to & H^1(\Gamma, E[n]) & \to & H^1(\Gamma, E \\
 & \downarrow & & \downarrow & & \downarrow \\
0\to & \prod E(\mathbb{Q}_p)/n & \to & \prod H^1(\Gamma_p, E[n]) & \to & \prod H^1(\Gamma_p,
\end{array}$$

Define

$$Sha(E) := Ker(H^1(\Gamma, E) \to \prod H^1(\Gamma_p, E))$$

and

$$Sel_n(E) \subset H^1(\Gamma, E[n])$$

to be the inverse image of $Sha(E)[n] \subset H^1(\Gamma, E)[n]$.

Get an exact sequence

$$0 \to E(\mathbb{Q})/n \to Sel_n(E) \to Sha(E)[n] \to 0$$

Fact:

$$Sel_n(E)$$

is finitely generated and computable. Computability depends on the finite generation and the possibility of checking *local* triviality of torsors, i.e., local existence of solutions.

Conjecture(BSD):

$Sha(E)$ is finite.

Possibly the most important part of BSD.

Run over powers of a single prime.

Then

$$\begin{array}{ccccccccc} 0\to & E(\mathbb{Q})/l^n & \to & Sel_{l^n}(E) & \to & Sha(E)[l^n] & \to 0 \\ & \downarrow & & \downarrow l^{n-1} & & \downarrow l^{n-1} & \\ 0\to & E(\mathbb{Q}/l & \to & Sel_l(E) & \to & Sha(E)[l] & \to 0 \end{array}$$

Assuming BSD, the map

$$Sha(E)[l^n]\to Sha(E)[l]$$

is zero for large n, and $E(\mathbb{Q})/l$ can be identified in $Sel_l(E)$ with the image of $Sel_{l^n}(E)$.

So assuming BSD, eventually can compute the elusive

$$\text{rank}E(\mathbb{Q}).$$

Rephrasing of this idea: Assuming BSD,

$$E(\mathbb{Q}) \otimes \mathbb{Z}_l \simeq H^1_f(\Gamma, T_l E)$$

where

$$H^1_f(\Gamma, T_l E) := \varprojlim Sel_{l^n}(E)$$

and

$$H^1_f(\Gamma, T_l E)$$

is computable.

In general Galois cohomology

$$H^1(\Gamma, T_l E)$$

can be interpreted as equivalence classes of extensions

$$0 \to T_l E \to M \to \mathbb{Z}_l \to 0$$

Note that from a point of E, we get such an extension for each l, which furthermore satisfy certain local conditions (corresponding to $H^1_f(\Gamma, T_l E)$).

Over the complex numbers, we also have

$$E(\mathbb{C}) \simeq H^0(\Omega_E)^*/H_1(E,\mathbb{Z}) = F^0\backslash H_1(E,\mathbb{C})/H_1(E,\mathbb{Z}$$

and the later object classifies extensions of Hodge structures

$$0 \rightarrow H_1(E,\mathbb{Z}) \rightarrow M \rightarrow \mathbb{Z} \rightarrow 0$$

That is, such extensions occur uniformly in all cohomology theories associated to E starting from a point of E.

General conjecture:

$$E(\mathbb{Q}) \otimes \mathbb{Q} \simeq Ext^1_{MMot_{\mathbb{Z}}}(\mathbb{Q}, H_1(E))$$

where the extensions occur in the category of mixed motives over $\mathbb{Z}$.

Cannot make this conjecture precise at present.

But statements like

$$E(\mathbb{Q}) \otimes \mathbb{Q}_l \simeq H^1_f(\Gamma, T_l E) \otimes \mathbb{Q}_l$$

should be just a 'realization' of this underlying formula.

Several important generalizations of the finiteness of Sha occur in the the context of Bloch-Kato conjecture and Fontaine-Mazur conjecture. One generally defines

$$H^1_f(\Gamma, V)$$

(or variations of it which we will denote in the same way) for motivic Galois representations V.

In Fontaine-Mazur, the statement becomes:

If V is motivic, then all extensions in $H^1_f(\Gamma, V)$ are motivic.

In Bloch-Kato, the corresponding conjecture is the surjectivity of a p-adic Chern class map:

$$K^{(r)}_{2r-n-1}(V) \otimes \mathbb{Q}_p \rightarrow H^1_g(\mathbb{Q}, H^n(\bar{V}, \mathbb{Q}_p(r)))$$

For a general smooth projective variety X of dimension d, if we fix a point $b \in X(\mathbb{Q})$, then for any other point x, the exact sequence

$$0 \rightarrow H^{2d-1}(X)(d) \rightarrow H^{2d-1}(X \setminus \{b, x\})(d) \rightarrow \mathbb{Q} \rightarrow 0$$

should define a map

$$X(\mathbb{Q}) \rightarrow Ext^1(\mathbb{Q}, H^{2d-1}(X)(d))$$

useful for studying the Diophantine geometry of $X(\mathbb{Q})$.

Meanwhile, we should have:

$$ord_{s=1}L(H^1(X), s) = \mathrm{dim}Ext^1(\mathbb{Q}, H^{2d-1}(X)(d))$$

$$= \mathrm{dim}(CH^d(X))^0 \otimes \mathbb{Q}$$

These conjectures also generalize to incorporate other degrees of cohomology.

For example, in the case of $L(H^i(X), s)$ where i is odd, put $m = (i+1)/2$.

Then it is conjectured that

$$ord_{s=m} L(H^i(X), s) = \text{dim} CH^m(X)^0 \otimes \mathbb{Q}$$

where $CH^m(X)^0$ is the group of rational equivalence classes of codimension m algebraic cycles that are homologically equivalent to zero.

Very grand and difficult conjecture. When all conjectures are proved expect many significant applications thereof.

Part II.

Notice that all Diophantine invariants occurring in motivic conjectures are *abelianized*.

For example, if X has dimension d, then $CH^d(X)$ is roughly like the free abelian group generated by $X(\mathbb{Q})$.

Suppose X is an algebraic curve of genus ≥ 2. In this case,

$$\zeta(X, s) = \zeta(s)\zeta(s-1)/L(H^1(X), s)$$

One of the conjectures says

$$ord_{s=1} L(H^1(X), s) = \text{rank} J_X(\mathbb{Q})$$

where J_X is the Jacobian of X.

In this case,

$$L(H^1(X), s) = L(H^1(J_X), s)$$

so the statement is nothing but the BSD conjecture for the abelian variety J_X.

Far from being able to prove such statement.

But even were it to be true, would say nothing about the important set

$$X(\mathbb{Q})!!$$

Problem is the abelianization:

$$X(\mathbb{Q}) \hookrightarrow Ext^1(\mathbb{Q}, H^1(X)(1))$$

puts $X(\mathbb{Q})$ into an *abelian group* because the category $MMot$ is abelian.

The abelian group structure makes

$$Ext^1(\mathbb{Q}, H^1(X)(1))$$

easier to study than the 'sructureless' set $X(\mathbb{Q})$, but makes it much larger, i.e., $J_X(\mathbb{Q})$.

Goes back to the big disappointment in Weil's thesis:

Study of $J_X(\mathbb{Q})$ says little about $X(\mathbb{Q})$.

(Weil wanted to prove the Mordell conjecture.)

Need for a 'nonabelian' version of the Albanese map related to a non-abelian category.

Weil's fantasy (1938, 'Generalization of abelian functions'):

Importance of developing 'non-abelian mathematics.'

Ingredients should involve moduli of vector bundles and fundamental groups.

Weil thought such theories should have application to arithmetic. Plausible that he had the Mordell conjecture in mind.

Weil's paper began the theory of vector bundles on curves, leading eventually to Narashimhan-Seshadri, Donaldson, Simpson, etc.

No arithmetic theory of π_1 at the time.