Isometries in two and three dimensions

Given an inner product space $(V, <\cdot,\cdot>)$, a linear map $L : V \to V$ is an *isometry* if

$$< Lv, Lw > = < v, w >$$

for all $v, w \in V$. Of course, then $||Lv|| = ||v||$ for all v. But the converse is also true: If $||Lv|| = ||v||$ for all v, then $< Lv, Lw > = < v, w >$ for all $v, w \in V$. The proof is an easy exercise. Thus, isometries are exactly those linear transformations that preserve the lengths of vectors.

A simple example in $\mathbb{R}^2$ is the map

$$R_t : \mathbb{R}^2 \to \mathbb{R}^2$$

that rotates each vector around the origin counter-clockwise through an angle of t. Clearly, lengths of vectors are preserved so R_t is an isometry. If we compute the matrix with respect to the standard basis, then

$$R_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

If you've never seen this before, you should completely convince yourself now by drawing a picture. That is, what are $R_t((1,0)^T)$ and $R_t((0,1)^T)$?

In fact, if $B = \{b_1\, b_2\}$ is *any* orthonormal basis with b_2 pointing at right angles to b_1 in a counter-clockwise direction, then we get the the same matrix when computed with respect to B. It is easy to check that

$$R_t^T R_t = R_t R_t^T = I,$$

confirming algebraically that $R_t^T = R_t^{-1}$. But there is one more evident condition satisfied by R_t. We have

$$\det(R_t) = \cos^2(t) + \sin^2(t) = 1$$

This determinant condition is not satisfied by all isometries. For example, if S just switches the order of the coordinates,

$$S(v_1, v_2) = (v_2, v_1),$$

then it is obviously an isometry, but $\det(S) = -1$. Of course we already know that for an isometry L of $\mathbb{R}^n$, we have

$$1 = \det(I) = \det(L^T L) = \det(L)^2$$

so that $\det(L) = \pm 1$.

In the dimension two, we have:

Theorem 1 *Suppose an isometry L has determinant one. Then L is a rotation. That is, there is some t for which*

$$L = R_t.$$

Proof.

Write the matrix for L as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since the columns are orthonormal, we have

$$a^2 + c^2 = b^2 + d^2 = 1$$

Denoting by t the angle between the positive x-axis and the vector $(a,c)^T$ measured in the counter-clockwise direction, we know therefore that

$$(a,c)^T = (\cos(t), \sin(t))^T$$

Since the vector $(b,d)^T$ is orthogonal to it and of length one, the only possibilities are $\pi/2$ either counter-clockwise or clockwise from $(a,c)^T$, which would correspond respectively to

$$(\cos(t+\pi/2), \sin(t+\pi/2))^T$$

or

$$(\cos(t-\pi/2), \sin(t-\pi/2))^T = -(\cos(t+\pi/2), \sin(t+\pi/2))^T$$

Standard trigonometric identities give

$$\cos(t+\pi/2) = \cos(t)\cos(\pi/2) - \sin(t)\sin(\pi/2) = -\sin(t)$$

$$\sin(t+\pi/2) = \sin(t)\cos(\pi/2) + \sin(\pi/2)\cos(t) = \cos(t)$$

Thus, in the first case, we would get $L = R_t$ as desired. But in the second case, the matrix would be

$$L = \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}$$

leading to $\det(L) = -1$, contrary to our assumption. □

We have characterized the rotations as being exactly the isometries of determinant one.

Somewhat surprisingly, this theorems extends to dimension 3. Of course to define a rotation in three dimensions, we need to choose an axis first, in fact, a directed axis (if we want to know what's meant by 'counter-clockwise'). So let l be such a directed axis, that is, a line through the origin with an arrow running in one direction. Define R^l_t to be the rotation around l through an angle t, in a counter-clockwise direction when we look down at the head of the arrow.

Lemma 2

$$\det(R^l_t) = 1$$

Proof. To see this, choose a basis for $\mathbb{R}^3$ where the first element b_1 goes in the direction of l, and then second and third element are orthonormal inside the plane perpendicular to l with b_3 being $\pi/2$ counterclockwise from b_2, again looking at the head of the arrow. Evidently, R^l_t does nothing to b_1 and preserves the plane. In fact, the matrix of R^l_t with respect to this basis is clearly

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

giving us $\det(R^l_t) = 1$. □

You should note the power of the fact that the determinant can be computed using any basis. Computing the matrix of R^l_t directly in the standard basis would be quite painful.

The interesting fact is the converse statement.

Theorem 3 *Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry of determinant 1. Then*

$$L = R^l_t$$

for some directed line l and some t.

Proof. The key point is the algebraic fact:

$$\det(L - I) = 0$$

This follows from

$$\det(L - I) = \det(L - LL^T) = \det(L(I - L^T)) = \det(L)\det(I - L^T)$$
$$= \det(I - L^T) = \det(I - L)$$

on the one hand, and on the other,

$$\det(I - L) = -\det(L - I)$$

since the matrix is (3×3). So we have no choice but $\det(L - I) = 0$.□

The significance here is that there must be a vector v such that $(L - I)v = 0$ or $Lv = v$, which then can be used to span a directed axis. That is, if W is the plane orthogonal to v, then $L(W)$ is still orthogonal to v, since L preserved inner products. Hence, we can speak of the restriction $L|W$, which is an isometry of the plane. If we choose any basis $\{b_2, b_3\}$ for W and compute the matrix of L with respect to $\{v, b_2, b_3\}$, then it has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

with the lower right (2×2) block being the matrix of $L|W$, which then implies that

$$\det(L|W) = \det(L) = 1$$

In addition, $L|W$ is clearly an isometry, since it still preserves the lengths of vectors. Therefore, by the previous theorem, $L|W$ is a rotation. If we now restrict b_2 and b_3 to be an orthonormal basis as before, then the matrix of L is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

for some t. That is to say, L is a rotation R^l_t, where l is the line in the direction of v.□

This leads to the rather striking

Corollary 4 *A composition of two rotations is a rotation.*

This result would be obvious in two dimensions, but I've never been able to successfully visualize it in 3D. The problem is that the rotations under discussion could be R^l_t and R^m_s with the axes l and m completely different. But we are saying that

$$R^l_t \circ R^m_s = R^n_u$$

for some third axis n and angle u. Geometric intuition (at least mine) doesn't seem to work too well, so we are resorting to a characterization of the rotations, namely, being an isometry of determinant one, that is evidently closed under composition. This is a common theme in many beautiful mathematical theorems, where you show that a set has some property by searching for a precise description of the set that makes the property obvious. In the present situation, it is the power of *algebraization*, whereby the geometric property of being an isometry is translated into the algebraic condition

$$L^T = L^{-1}$$

and then the curious property of (odd)$\times$(odd)-determinants comes into play.

We close by mentioning that if

$$L : \mathbb{R}^2 \to \mathbb{R}^2$$

is an isometry of determinant -1, then $S \circ L$ has determinant $\det(S)\det(L) = 1$, and hence, equals R_t for some t. Since $S = S^{-1}$, we get that $L = S \circ R_t$. In words, an arbitrary isometry is either a rotation, or a rotation followed by a flip of coordinates.

Similarly, if

$$L : \mathbb{R}^3 \to \mathbb{R}^3$$

is an isometry of determinant -1, then we can utilize

$$S_{12}(x, y, z) = (y, x, z)$$

in the same way to conclude that an isometry in $\mathbb{R}^3$ is either a rotation, or a rotation followed by a flip of the (x, y)-coordinates.

Here is an exercise that is surprising easy: Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is *any* map that preserves distances. That is, we are not requiring f to be even linear. Show that $f = T_v \circ L$ where L is a linear isometry, and T_v, for a given vector v, is the map that translates any other w by v:

$$T_v(w) = w + v.$$

You see, we should probably have used the terminology 'isometry' to refer to any distance-preserving map, and called the linear transformations that preserve norms *linear isometries* as I have done just now. But the abuse of language was not so bad because the two kinds end up merely differing by translations. Another version of this statement is that these more general isometries reduce to linear isometries if they are just required to *fix the origin.*