Classical Motives I: Motivic L-functions

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Variety $V \rightsquigarrow \zeta(V,s)$, ζ-function of V

$\rightsquigarrow$ arithmetic invariants of V.

This picture should be refined by

Variety $V \rightsquigarrow$ constituent motives $\{M_i\}$

$\rightsquigarrow \{\ L(M_i,s)\}$, L-functions of the M_i

$\rightsquigarrow$ arithmetic invariants of the M_i

$\rightsquigarrow$ arithmetic invariants of V.

Example:

$E/\mathbb{Q}$ elliptic curve with affine minimal equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

S: set of primes of bad reduction.

$\mathcal{E}$: proper smooth model of E over $\mathbb{Z}[1/S]$.

$(\mathcal{E})_0$: set of closed points.

$$\zeta_S(E,s) = \prod_{x\in(\mathcal{E})_0} \frac{1}{1-N(x)^{-s}}$$

Then there is a decomposition

$$\zeta_S(E,s) = \zeta_S(s)\zeta_S(s-1)/L_S(E,s)$$

where

$$\zeta_S(s) = \prod_{p \notin S} \frac{1}{1 - p^{-s}}$$

and

$$L_S(E, s) = \prod_{p \notin S} L_p(E, s)$$

is the partial L-function of E with factors defined by

$$L_p(E, s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

Here $a_p = p + 1 - N_p$ and N_p is the number of points on E mod p.

Can put in bad Euler factors according to a recipe determined by the reduction of E at p:

$$L_p(s) = \begin{cases} 1/(1 - p^{-s}) & \text{split multiplicative;} \\ 1/(1 + p^{-s}) & \text{non-split multiplicative;} \\ 1 & \text{additive.} \end{cases}$$

$$L(E, s) := \prod_p L_p(E, s)$$

Using thus the breakdown into three factors, we can also complete $\zeta_S(E, s)$ in a natural way.

The estimate $|a_p| \leq 2\sqrt{p}$ implies that the Euler product converges for $Re(s) > 3/2$.

To control the analytic properties, use relation to automorphic L-functions.

In this case, can make explicit by computing the conductor

$$N_E := \prod_{p \in S} p^{f_p}$$

Here

$$f_p = ord_p(\Delta_E) + 1 - m_E$$

where Δ_E is the discriminant of E and m_E is the number of components over $\bar{\mathbf{F}}_p$ of a Neron model of E.

Fact (W, T-W, BCDT): L has an analytic continuation to the complex plane.

In fact,

$$L(E,s) = L(f_E, s)$$

$$= \frac{1}{(2\pi)^s \Gamma(s)} \int_0^\infty f_E(iy) y^{s-1} dy$$

for a normalized weight 2 new cusp form f_E of level N_E which is an eigenvector for the Hecke operators, determined by a q expansion

$$f_p = 1 + a_1 q + a_2 q^2 + \cdots$$

where the a_p have to be the same as those for E when $p \notin S$.

Can find f_E and then use this formula for compute L-values.

Conjecture (BSD):

$$ord_{s=1}L(E,s) = \text{rank}E(\mathbb{Q})$$

Proved if $ord_{s=1}L(E,s) \leq 1$. (Kolyvagin)

Functional equation:

$$\Lambda(E,s) := (2\pi)^s \Gamma(s) N_E^{s/2} L(E,s)$$

satisfies a functional equation

$$\Lambda(E, 2-s) = \epsilon_E \Lambda(E,s)$$

where $\epsilon_E = \pm 1$ depends on the curve E. Can be computed in a straightforward way as a product of local terms.

Now, if $\epsilon_E = -1$, then clearly

$$L(E,1) = 0$$

Suppose you can check $L'(E,1) \neq 0$ using the equality with $L'(f,1)$, then we conclude $E(\mathbb{Q})$ has rank one.

Thus, analysis of the L-function, including the functional equation and computation, gives us the structure of $E(\mathbb{Q})$.

Continuation of BSD: If r is the order of vanishing, then

$$(s-1)^{-r}L(E,s)|_{s=1}$$

$$= |Sha(E)|R_E\Omega\prod_p c_p/|E(\mathbb{Q})(tor)|^2$$

relating L-values to many other *refined arithmetic invariants* of E.

General principle: L-function encodes Diophantine invariants of E.

Brief discussion of terms.

Important distinction: Rational terms versus transcendental terms.

Rational terms:

$Sha(E)$: The Tate-Shafarevich group of E, conjectured to be finite. Classifies locally trivial torsors for E. Analogous to a class group.

$E(\mathbb{Q})(tor)$: (finite) torsion subgroup of $E(\mathbb{Q})$.

c_p: Tamagawa number.

$$c_p = (E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p))$$

Transcendental terms:

R_E: Regulator of E computed using canonical height $<,>$ and basis $\{P_1, P_2, \ldots, P_r\}$ for $E(\mathbb{Q})/E(\mathbb{Q})(tor)$.

$$R_E := |\det(< P_i, P_j >)|$$

Thus, R_E is the covolume of the Mordell-Weil group, in a manner analogous to the classical regulator of number fields (covolume of units).

Ω: real period

$$\Omega = \int_\gamma |\omega|$$

where

$$\omega = dx/(2y + a_1 x + a_3)$$

and

$$< \gamma >= H_1(E(\mathbb{C}), \mathbb{Z})^+$$

The known relations between L-functions and arithmetic are expected to generalize vastly.

L-functions defined using Galois actions on étale cohomology and completed using Hodge theory.

Conjectures:.

(1) Hasse-Weil: analytic continuation and functional equation, addressed by Langlands' program: 'Motivic L-functions are automorphic L-functions.'

(2) Values:

(a) Deligne generalizes discussion of period (in non-vanishing case) using comparison of rational De Rham and topological cohomologies;

(b) Beilinson-Bloch generalizes discussion of order of vanishing and regulator using rank and covolume of motivic cohomology.

(c) Bloch-Kato generalizes discussion of rational part using Tamagawa numbers for Galois representations via p-adic Hodge theory.

$X/\mathbb{Q}$: smooth projective variety.

Associated to X is a collection of cohomology groups, the *realizations* of the motive of X.

$H_l^n(X) = H_{et}^n(\bar{X}, \mathbb{Q}_l)$ for each prime l: the $\mathbb{Q}_l$-coefficient étale cohomology of degree n. Carries a natural action of $\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

$H_{DR}^n(X) := H^n(X, \Omega^{\cdot}_X)$: the algebraic De Rham cohomology equipped with a Hodge filtration given by

$$F^i H_{DR}^n(X) = H^n(X, \Omega^{\geq i}) \hookrightarrow H_{DR}^n(X)$$

for each i.

$H_B^n(X) := H^n(X(\mathbb{C}), \mathbb{Q})$: the $\mathbb{Q}$-coefficient singular cohomology of the complex manifold $X(\mathbb{C})$ equipped with a continuous action F_∞ of complex conjugation.

The completed L-function of $H^n(X)$ uses all these structures.

Canonical comparison isomorphisms:

$$H^n_B(X) \otimes \mathbb{Q}_l \simeq H^n_l(X)$$

preserving action of F_∞.

$$H^n_B(X) \otimes \mathbb{C} \simeq H^n_{DR}(X) \otimes \mathbb{C}$$

This isomorphism endows $H^n_B(X)$ with a *rational Hodge structure* of weight n 'defined over $\mathbb{R}$.'

That is, we have a direct sum decomposition

$$H_B^n(X) \otimes \mathbb{C} \simeq \oplus H^{p,q}(X)$$

where

$$H^{p,q} := F^p \cap \bar{F}^q$$

and

$$F_\infty(H^{p,q}) = H^{q,p}$$

If we denote by c the complex conjugation on $\mathbb{C}$ then

$$(H_B^n(X) \otimes \mathbb{C})^{F_\infty \otimes c} = H_{DR}^n \otimes \mathbb{R}$$

At non-archimedean places, there is an important analogue.

For any embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$, we have

$$D_{DR}(H^n_l(X)) := (H^n_l(X) \otimes B_{DR})^{\Gamma_l}$$

$$\simeq H^n_{DR}(X) \otimes \mathbb{Q}_l$$

where $\Gamma_l = \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$, and B_{DR} is Fontaine's ring of p-adic periods.

Regardless of its precise definition, a motive M should have associated to it a collection of objects as above that we call *a pure system of realizations* that make up a category $\mathcal{R}$.

That is, this is a collection

$$R(M) = \{\{M_l\}, M_{DR}, M_B\}$$

where each M_l is a representation of Γ on a (finite-dimensional) $\mathbb{Q}_l$-vector space, M_{DR} is a filtered $\mathbb{Q}$-vector space, and M_B is a $\mathbb{Q}$-vector space with an involution F_∞. These vector spaces should all have the same dimension and be equipped with a system of comparison isomorphisms as above.

This data must be subject to further constraints having to do with local Galois representations.

Recall exact sequence:

$$0 \to I_p \to \Gamma_p \xrightarrow{v} \hat{\mathbb{Z}} \to 0$$

where I_p is the inertia group and

$$\hat{\mathbb{Z}} \simeq \mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p).$$

$Fr_p \in \hat{\mathbb{Z}}$ corresponds to the geometric Frobenius, that is, the inverse to the p-power map.

For $l \neq p$, I_p has a tame l-quotient

$$t_l : I_p \to I_{p,l}$$

with the structure

$$I_{p,l} \simeq \hat{\mathbb{Z}}_l(1) \simeq \varprojlim \mu_{l^n}$$

as a module for $\mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$.

Define

$$W_p := v^{-1}(\mathbb{Z}) \subset \Gamma_p,$$

the *Weil group* at p.

Convenient to analyze the data of M_l using an associated *Weil-Deligne* (W-D) representation

$$WD_p(M_l)$$

for each p, consisting of

-a representation r of W_p such that $r|I_p$ has finite image,

-and a nilpotent operator N_p acting on the representation.

These satisfy a compatiblity

$$r(\phi_p)N_p r(\phi_p^{-1}) = p^{-1}N_p$$

for any lift $\phi_p \in W_p$ of Fr_p.

The construction of $WD_p(M_l)$ for $p \neq l$ uses the fact that the action of I_p when restricted to some finite index subgroup I'_p is unipotent, and hence, can be expressed as

$$\sigma \mapsto \exp(t_l(\sigma) N_p)$$

for a nilpotent N_p. Then the representation r is given by

$$r(\phi_p^n \sigma) = \phi_p^n \sigma \exp(-t_l(\sigma) N)$$

For $p = l$, we use the fact that any De Rham representation is potentially semistable, and hence, gives us a filtered (ϕ_l, N_l) module via

$$M_l \mapsto (M_l \otimes B_{st})^{\Gamma'_l}$$

which is, in any case, isomorphic to M_{DR}.

Remarks:

-The point of this construction is that we can package the information of the representation in a form that does not use the topology of $\mathbb{Q}_l$. Thereby makes natural the connection to complex automorphic forms.

-Creates a precise analogy with limit mixed Hodge structures.

-We can define the

Frobenius semi-simplification $WD_p(M_l)^{ss}$

of $WD_p(M_l)$ by replacing ϕ_p with its semi-simple part.

Here are the constraints we impose on our pure system of realizations:

-We assume then that there exists a finite set S of primes such that $WD_p(M_l)$ is unramified for all $p \notin S$, i.e., $N_p = 0$ and I_p acts trivially.

-'Algebraicity and independence of l':

There exists a Frobenius semi-simple W-D representation $WD_p(M)$ over $\bar{\mathbb{Q}}$ such that

$$WD_p(M) \otimes \bar{\mathbb{Q}}_l \simeq WD_p^{ss}(M_l) \otimes \bar{\mathbb{Q}}_l$$

for any embedding

$$\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$$

Subject to these conditions, the collection $\{M_l\}$ is then referred to as a *strongly compatible* system of l-adic representations.

-'Weil conjecture':

There should exist an integer n, called the *weight* of M, such that the eigenvavlues of Fr_p acting on $WD_p(M)$ for $p \notin S$ have all Archimedean absolute values equal to $p^{n/2}$. Furthermore, the Hodge structure M_B should be pure of weight n.

-'purity of monodromy filtration': If we denote by $Mn_.$ the unique increasing filtration on $WD_p(M)$ such that $Mn_{-k} = 0$, $Mn_k = WD_p(M)$ for sufficiently large k and

$$N(Mn_k) \subset Mn_{k-2},$$

then the associated graded piece

$$Gr_k^{Mn}(WD_p(M))$$

has all Frobenius eigenvalues of archimedean absolute value $p^{(n+k)/2}$.

Remarks:

-In general, need to allow coefficients in E_λ for the representations where E is a number field and E_λ are completions. Arise naturally when considering direct summands or *motives with coefficients*, e.g., abelian varieties with CM.

-The bi-grading

$$M_B \otimes \mathbb{C} \simeq \oplus M^{p,q}$$

which is compatible with the complex conjugation of coefficients corresponds to a representation of the group

$$Res_{\mathbb{R}}^{\mathbb{C}}(\mathbf{G}_m)$$

-Together with the action of

$$F_\infty \circ C$$

it can be viewed as a representation of the real Weil group with points given by

$$W_{\mathbb{R}}(\mathbb{R}) = \mathbb{C}^* \cup \mathbb{C}^* j$$

where $j^2 = -1$ and $jzj^{-1} = \bar{z}$.

Here, C is the Weil operator defined by

$$C|M^{pq} = i^{q-p}$$

-It is conjectured that the realizations

$$H^n(X) = (\{H^n_l(X)\}, H^n_B(X), H^n_{DR}(X))$$

coming from a smooth projective variety X satisfy the algebraicity, independence of l, and purity conditions even for $p \in S$.

Category of pure motives should be comprised of objects in $\mathcal{R}$ of *geometric origin*, a notion with a rather precise interpretation. For example, need to allow duals (homology) and tensor products of all objects considered.

Objects that are not generated in an obvious way from those of the form

$$H^n(X)$$

arise via images (or kernels) under pull-backs and push-forwards in cohomology induced by maps of varieties, as well as $\mathbb{Q}$-linear combinations of geometric maps.

Also should be able to compose pull-backs with pushforwards.

Such compositions give rise to the idea of using *correspondences* modulo homological equivalence as morphisms.

Once morphisms are constructed in this manner, get naturally new objects using the decomposition of

$$End(H^n(X)),$$

which is a semi-simple $\mathbb{Q}$-algebra subject to one of the standard conjectures that numerical equivalence and homological equivalence coincide.

Can consider a category of mixed systems of realizations by requiring a weight filtration

$$\cdots \subset W_{n-1}M \subset W_nM \subset W_{n+1}M \subset$$

compatible with all the comparisons and such that each graded quotient

$$Gr^n_W(M)$$

is a pure system of realizations of weight n.

Mixed motives should be those of geometric origin such as the cohomology of varieties that are not necessarily smooth or proper. But then, need to include objects like (finite-dimensional quotients of)

$$\mathbb{Q}[\pi_1]$$

or the (co)-homology of (co-)simplicial varieties.

Given a pure system M of realizations we can define its L-function $L(M,s)$ as an Euler product

$$L(M,s) = \prod_{p} L_p(M,s)$$

with

$$L_p(M,s)$$

$$= \frac{1}{\det[(1 - p^{-s} Fr_p)|(WD_p(M))^{I_p=1, N_p=0}]}$$

Assume M is of weight n, then product converges (and hence is non-zero) for

$$Re(s) > n/2 + 1.$$

Also a factor at ∞ depending upon the representation $M_B \otimes \mathbb{C}$ of $W_{\mathbb{R}}$.

Define

$$\Gamma_{\mathbb{R}} := \pi^{-s/2}\Gamma(s/2)$$

$$\Gamma_{\mathbb{C}} := 2(2\pi)^{-s}\Gamma(s)$$

$$h^{pq} := \dim M^{pq}$$

$$h^{p,\pm} := \dim M^{pp,\pm 1}$$

where the signs in the superscript refer to the ± 1 eigenspaces of the F_∞-action.

Then

$$L_\infty(M,s)$$

is defined by

$$\mathop{}_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{pq}}$$

for odd n, and

$$\mathop{}_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{pq}} \Gamma_{\mathbb{R}}(s-n/2)^{h^{n/2+}} \Gamma_{\mathbb{R}}(s-n/2+1)^{h^{n/2-}}$$

for n even.

It is conjectured that $\Lambda(M,s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(M^*,1-s)$$

where the epsilon factor has the form $\epsilon(M,s) = ba^s$. This conjecture should be addressed by the Langlands' program.

Notation:

$\mathbb{Q}$: trivial system of realizations.

$\mathbb{Q}(1) := H^2(\mathbf{P}^1)^*$

$\mathbb{Q}(i) = \mathbb{Q}(1)^{\otimes i}$ for $i \geq 0$

and $\mathbb{Q}(i) = \mathrm{Hom}(\mathbb{Q}(-i), \mathbb{Q})$ for $i < 0$.

For a system M of realizations,

$$M(i) := M \otimes \mathbb{Q}(i)$$

Then for any smooth projective variety of dim d, we have

$$H^{2d}(X) \simeq \mathbb{Q}(-d)$$

and a perfect pairing

$$H^i(X) \times H^{2d-i}(X) \rightarrow H^{2d}(X)$$

Cup product with the cohomology class of a hyperplane gives us

$$H^i(X) \simeq H^{2d-i}(X)(d-i)$$

Some simple properties of twisting:

$M(n)_l$ is the tensor product of M_l with the n-th power of the $\mathbb{Q}_l$ cyclotomic character.

$$F^i(M(n)_{DR}) = F^{n+i}M_{DR}$$

with a corresponding shift in Hodge numbers h^{pq}.

$$F_\infty|M(n)_B = (F_\infty|M_B) \otimes (-1)^n$$

Finally,

$$L(M(n), s) = L(M, s+n)$$

Henceforward, we will focus on the case where M is $H^n(X)$ for a smooth projective variety X of dimension d and assume that

- $H^n(X)$ is a pure system of realizations;

- the analytic continuation and functional equation hold true.

Conjectures on orders.

We have

$$H^n(X)^* \simeq H^{2d-n}(X)(d) \simeq H^n(X)(n)$$

Thus, the functional equation relates

$$L(H^n(X), s)$$

and

$$L(H^n(X)(n), 1-s) = L(H^n(X), n+1-s)$$

with center of reflection

$$(n+1)/2$$

Thus, for the most part, we can confine interest to

$$m \geq (n+1)/2$$

or, equivalently,

$$n+1-m \leq (n+1)/2.$$

Brief reminder on two simple case:

n odd: BSD

$$ord_{s=1}L(H^1(E), s) = \text{rank}E(\mathbb{Q})$$

Now, an element

$$x \in E(\mathbb{Q})$$

gives rise to an extension in the category $\mathcal{R}$ of realizations

$$\delta(x) \in Ext^1_{\mathcal{R}}(\mathbb{Q}, H^1(E)(1))$$

It is conjectured that when $\mathcal{R}$ is replaced by a suitable category of motives, this is the only way to construct such extensions.

n even:

$F/\mathbb{Q}$ Galois extension and

$$\rho : \text{Gal}(F/\mathbb{Q}) \to Aut(V)$$

a finite-dimensional representation.

$$L(\rho, s)$$

Artin L-function. Then

$$ord_{s=1} L(\rho, s) = -\text{dimHom}_{Rep}(\mathbb{Q}, V)$$

The general conjecture is

$$ord_{s=n+1-m}L(H^n(X),s)$$

$$= \text{dim}Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m))$$

$$-\text{dimHom}_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m))$$

The Hom and Ext should occur inside a conjectural category of mixed motives over $\mathbb{Z}$ with $\mathbb{Q}$-coefficients.

For weight reasons, the Hom term vanishes unless $n = 2m$ in which case the Ext term vanishes. That is, in the pure situation we are considering, only one term or the other occurs.

This is the prototype of the sort of statement that should hold for an arbitrary (mixed) motive.

So when $n = 2m$, this becomes

$$ord_{s=m+1}L(H^{2m}(X), s) =$$

$$-\text{dimHom}_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m}(X)(m))$$

generalizing he pole of the Artin L-function ($m = 0$).

It is expected that

$$\text{Hom}_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m}(X)(m))$$

$$\simeq [CH^m(X)/CH^m(X)^0] \otimes \mathbb{Q}$$

Of course the isomorphism should arise via a cycle map

$$CH^m(X) \to H^{2m}(X)(m)$$

killing the cycles $CH^m(X)^0$ homologically equivalent to zero.

When $n+1 = 2m$, the conjecture predicts the order of vanishing at the central critical point:

$$ord_{s=m} L(H^{2m-1}(X), s)$$

$$= \text{dim} Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m-1}(X)(m))$$

It is then conjectured that

$$\text{dim} Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m-1}(X)(m)) \simeq CH^m(X)^0 \otimes \mathbb{Q}$$

The map from cycles to extensions goes as follows: given a representative Z for a class in $CH^m(X)^0$, we get an exact sequence

$$0 \to H^{2m-1}(X)(m) \to H^{2m-1}(X \setminus Z)(m)$$

$$\xrightarrow{\delta} H_Z^{2m}(X)(m) \to H^{2m}(X)(m)$$

There is a local cycle class

$$cl(Z) \in H_Z^{2m}(X)(m)$$

that maps to zero in $H^{2m}(X)(m)$, giving rise to the desired extension:

$$0 \to H^{2m-1}(X)(m) \to \delta^{-1}(cl(Z)) \to \mathbb{Q} \to 0$$

These two classical points, central critical:

$$n+1-m=m=(n+1)/2, \qquad n \text{ odd},$$

and just right of it:

$$n+1-m=n/2+1, \quad n \text{ even},$$

are somewhat exceptional. In all other cases, one expects

$$Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)) = H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m))$$

with the last group, often referred to as *motivic cohomology*, defined using K-theory :

$$Im[(K_{2m-n-1}(\mathcal{X}))^{(m)} \to (K_{2m-n-1}(X))^{(m)}]$$

($\mathcal{X}$ is a proper flat regular $\mathbb{Z}$-model for X) or Bloch's higher Chow groups

$$Im[CH^{n+1}(\mathcal{X}, 2m-n-1) \otimes \mathbb{Q}$$

$$\to CH^{n+1}(X, 2m-n-1) \otimes \mathbb{Q}]$$

Latter interpretation more popular lately.

However, intrinsic interpretation in terms of the category of motives should be kept in mind in all constructions.

In fact, when $m > n/2+1$, the conjectured functional equation implies

$$ord_{s=n+1-m} L(H^n(X), s)$$

$$= \mathrm{dim} Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

where the extension occurs inside the category of real mixed Hodge structures defined over $\mathbb{R}$. So the conjecture on order of vanishing follows from the conjecture that the Hodge realization functor induces an isomorphism

$$Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)) \otimes \mathbb{R}$$

$$\simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

In general, conjecture should be conceptualized in two parts:

(1) Relation between L functions and Ext groups in category of motives.

(2) Geometric interpretation of Ext groups.

Provides unity to a wide range of related issues in Diophantine geometry.

There is a construction, convenient in practice, of the real Ext group via Deligne cohomology:

$$Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

$$\simeq H^{n+1}_D(X_{\mathbb{R}}, \mathbb{R}(m))$$

and using properties of Deligne cohomology, one can construct regulator maps

$$H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m))$$

$$\to Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

that can be studied independently of a category of motives.

For example, can construct subgroups

$$L \subset H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)),$$

that should conjecturally be of full rank, and study their image.

Conjectures on transcendental part of values.

Central critical values (Bloch-Beilinson): $s = m$, $n = 2m - 1$.

We have an isomorphism

$$F^m H^{2m-1}_{DR}(X) \otimes \mathbb{R}$$

$$\simeq [H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R}$$

This is then realized as an isomorphism

$$[\wedge^{top}(F^m H^{2m-1}_{DR}(X))]^{-1} \otimes$$

$$\wedge^{top}[[H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}}] \otimes \mathbb{R}$$

$$\simeq \mathbb{R}$$

Choosing bases for the two $\mathbb{Q}$-lines determines a period

$$p(H^{2m-1}(X)(m)) \in \mathbb{R}^*/\mathbb{Q}^*$$

Get additional transcendental contribution by considering a *height pairing*, conjectured to be non-degenerate:

$$CH^m(X)^0 \times CH^{\dim(X)+1-m}(X)^0 \to \mathbb{R}$$

whose determinant gives us a regulator

$$r(H^{2m-1}(X)(m)) \in \mathbb{R}^*/\mathbb{Q}^*$$

Recall that conjecturally

$$d_m := ord_{s=m} L(H^{2m-1}(X), s)$$

$$= \dim CH^m(X)^0 \otimes \mathbb{Q}$$

As for the value then, it is conjectured that

$$L^*(H^{2m-1}(X), m)$$

$$:= \lim_{s \to m} (s-m)^{-d_m} L(H^{2m-1}(X), s)$$

$$= p(H^{2m-1}(X)(m)) r(H^{2m-1}(X)(m))$$

in $\mathbb{R}^*/\mathbb{Q}^*$.

Values at $n + 1 - m < n/2$.

Note that this is equivalent to

$$m > n/2 + 1,$$

the region of convergence for the Euler product. Hence, we are skipping the classically interesting case of

$$m = n/2 + 1 \qquad (n + 1 - m = n/2)$$

for n even.

Instead of a period isomorphism, there is then an exact sequence:

$$0 \to F^m H^n_{DR}(X) \otimes \mathbb{R} \to [H^n_B(X)(m-1)^{(-1)^{m-1}} \otimes \mathbb{R}]$$

$$\to Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R}) \to 0$$

Thus, the transcendental part should incorporate a $\mathbb{Q}$-structure on

$$Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

coming from the conjectured isomorphism

$$H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m))) \otimes \mathbb{R}$$

$$\simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

Assuming this, we are led to a trivialization

$$[\wedge^{top} F^m H^n_{DR}(X)]^{-1} \otimes \wedge^{top}([H^n_B(X)(m-1)]^{(-1)^{m-1}})$$

$$\otimes [\wedge^{top}(H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)))]^{-1} \otimes \mathbb{R}$$

$$\simeq \mathbb{R}$$

Thus, choosing bases for the three $\mathbb{Q}$-lines determines a number

$$c(H^n(X)(m)) \in \mathbb{R}^*/\mathbb{Q}^*.$$

Beilinson's conjecture is that

$$L^*(H^n(X), n+1-m)$$

$$= c(H^n(X)(m)).$$

For the value at $m = n/2$ $(n+1-m = n/2+1)$, the regulator incorporates maps both from motivic cohomology

$$H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m+1))$$

and

$$CH^m(X)^0.$$

In the Bloch-Kato conjecture isomorphisms are normalized more carefully, comparing certain integral structures one prime at a time. Thereby, it constructs a lift of $c(H^n(X)(m))$ to $\mathbb{R}^*$ and interprets

$$q(H^n(X)(m)) :=$$

$$L^*(H^n(X), n+1-m)/c(H^n(X)(m))$$

in terms of arithmetic invariants arising from Galois cohomology.

Extraction of the rational part is supposed to lead eventually to a p-*adic* L-function

$$\mathcal{L}^{(p)}(H^n(X))$$

that exercises control over Galois cohomology (i.e., Selmer groups) and Diophantine invariants.

Best strategy so far for 'direct application' of L-functions to the elucidation of Diophantine structures.

Warning: *conspicuous deficiency* in theory of motives:

Even in the best of possible worlds, only *abelian* invariants are accessible, such as

$$CH^m(X).$$

Does not yield information about

$$X(\mathbb{Q})$$

unless X is an abelian variety.

In fact, theory of motives is implicitly modelled after the theory of abelian varieties and H_1.

Attempts to address this deficiency for certain varieties are contained in

Grothendieck's *anabelian program*

that concerns itself with the theory of

pro-finite π_1's.

Also an interesting role for the intermediate theory of *motivic fundamental groups*, where Ext groups are replaced by *classifying spaces for non-abelian torsors*.