Primitive elements: an example

Just after answering a question this week in a slightly complicated way, I recalled that in the classes of examples that came up, the primitive element could be found just using the proof of the theorem concerning them. So I thought I'd illustrate this briefly. You should refer to the statement and proof in the web notes page 11.

Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. So we get the conjugates $\sqrt{3}, -\sqrt{3}$ and $\sqrt{5}, -\sqrt{5}$ as well. The theorem says the following: pick $c \in \mathbb{Q}$ so that $\sqrt{3} + c\sqrt{5}$ is not equal to $\sqrt{3} - c\sqrt{5}, \sqrt{3} - c\sqrt{5}, -\sqrt{3} + c\sqrt{5}$. Then $\sqrt{3} + c\sqrt{5}$ is a primitive element. Clearly $c = 1$ will do.

When you have more elements, say, $F = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$ you can find a primitive element by induction. One way is to write

$$F = \mathbb{Q}(\sqrt{3})(\sqrt{5}, \sqrt{7})$$

which we can then write as

$$\mathbb{Q}(\sqrt{3})(\alpha)$$

for some $\alpha \in \mathbb{Q}(\sqrt{3})$ and then proceed. If you follow the proof literally, this is the way the induction will go. But it's easier to do induction the other way, by writing $F = \mathbb{Q}(\sqrt{3}, \sqrt{5})(\sqrt{7})$ and finding the primitive element for $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ first. We've already done this of course, so that $F = \mathbb{Q}(\sqrt{3}+\sqrt{5}, \sqrt{7})$. The conjugates of $\sqrt{3} + \sqrt{5}$ are

$$\sqrt{3} + \sqrt{5}, \sqrt{3} - \sqrt{5}, -\sqrt{3} + \sqrt{5}, -\sqrt{3} + \sqrt{5}$$

while for $\sqrt{7}$, they are

$$\sqrt{7}, -\sqrt{7}$$

So we need to find $c \in \mathbb{Q}$ such that $\sqrt{3} + \sqrt{5} + c\sqrt{7}$ is not equal to

$$\sqrt{3} + \sqrt{5} - c\sqrt{7}, \sqrt{3} - \sqrt{5} - c\sqrt{7}, -\sqrt{3} + \sqrt{5} - c\sqrt{7}, -\sqrt{3} - \sqrt{5} - c\sqrt{7},$$

$$\sqrt{3} - \sqrt{5} + c\sqrt{7}, -\sqrt{3} + \sqrt{5} + c\sqrt{7}, -\sqrt{3} + \sqrt{5} + c\sqrt{7}$$

Again, we check that $c = 1$ works so that

$$\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7}) = \mathbb{Q}(\sqrt{3} + \sqrt{5} + \sqrt{7})$$

Of course the method appearing in example 37 also works. But computations of the sort that appear in example 25 (which is used) might occasionally be tedious, in which case the method of the proof can be easier.