Projective Geometry

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5 June 2008

Algebraic Construction

Consider the equation:

$$x^n + y^n = 1 \tag{1}$$

Let $x = a/c$ and $y = b/d$ be solutions of this equation, where a, b, c, d are all integers so that x and y are rational numbers in their lowest fraction term. Claim that $c = d$. Substituting solutions back into equation (1), we have

$$(a/c)^n + (b/d)^n = 1 \qquad \Rightarrow \qquad a^n d^n + b^n c^n = c^n d^n$$

It is clear that $c^n | a^n d^n$ and $d^n | b^n c^n$. Using the facts that $gcd(a, c) = 1$ and $gcd(b, d) = 1$, we can see that $c^n | d^n$ and $d^n | c^n$. So $c|d$ and $d|c \Rightarrow c = d$. Therefore, given any rational solution $(a/c, b/c)$ to (1), we can write and equation in the form $a^n + b^n = c^n$.
We can now rewrite (1) in it's *homogenised* form, which I shall write:

$$X^n + Y^n = Z^n \tag{2}$$

If, conversely, we have any integer solutions (a, b, c) for (2), we have automatically found solutions $(a/c, b/c)$ for (1).
Notice that solutions for (2) may lead to the same solutions for (1); (a, b, c) and (ta, tb, tc), t some integer, as solutions in (2) yield exact same solutions for (1). For now, think of (a, b, c) and (ta, tb, tc) as same solutions, this idea will become apparent once the projective plane is established.

By introducing (2), we have raised a potential problem. If n is odd, $(1, -1, 0)$ and $(-1, 1, 0)$ are both solutions to (2). What are their corresponding solutions in (1)? Infinity! Lets treat these solutions as "sensible" solutions at infinity.

The above set up allows us to define the *projective plane*. The idea is to take the triplet solutions as the coordinates on the plane. With respect to the projective plane, we choose to neglect the trivial solution $(0, 0, 0)$. Recall that (a, b, c) and (ta, tb, tc) are essentially same solutions, these correspond to *homogeneous coordinates* on the projective plane. That is, we can define an equivalence relation $\sim$ for a non-zero t such that if $[a, b, c] \sim [a', b', c']$ if $a = a't, b = b't, c = c't$.

Formally, the *projective plane* is

$$\mathbb{P}^2 = \frac{\{[a, b, c]\ ;\ a, b, c \text{ are not all zero}\}}{\sim}$$

This says if $[a, b, c] \sim [a', b', c']$, then $[a, b, c]$ and $[a', b', c']$ correspond to the same point on the projective plane.

Generally, using the nth dimensional equivalence relation, the *projective n-space* is defined

$$\mathbb{P}^n = \frac{\{[a_0, a_1, \ldots, a_n]\ ;\ a_0, a_1, \ldots, a_n \text{are not all zero}\}}{\sim}$$

A *line* in $\mathbb{P}^2$ is a set of points $[a, b, c]$ in $\mathbb{P}^2$ that satisfies

$$\alpha X + \beta Y + \gamma Z = 0$$

Geometric Construction

The general idea of this approach is to let *every* line in the plane to have *exactly* one intersection point. It is known that every pair of lines that are not parallel defines a unique intersection point. Now we could let every parallel line have an intersection point at infinity, and for each direction, there will be a *distinct* intersection point.

Define our usual Euclidean Plane as *affine plane*

$$\mathbb{A}^2 = \{(x, y)\ ;\ \ x, y \text{ are real numbers}\}$$

The projective plane can be defined as the union

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \{\text{the set of directions in } \mathbb{A}^2\}$$

Notes:

The points in $\mathbb{P}^2 \setminus \mathbb{A}^2$ are called *points at infinity.*
L_∞ is the line containing all the points at infinity.
Since there is an intersection for all pairs of distinct lines, intuitively speaking, the projective planes does not contain any parallel lines.

The direction of lines can be expressed by a line through the origin to some point $[A, B]$ on the plane. This allows us to also define

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$$

Both of the constructions above describe the projective plane correctly. In fact, one can produce a mapping between the two definitions such that the maps are inverses

$$\mathbb{P}^2 = \{[a,b,c] \ ; \ a,b,c \text{ are not all zero}\}/\sim \ \Leftrightarrow \ \mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$$

To put it simply, $c = 0$ corresponds to the projective line and $c \neq 0$ corresponds to the affine plane.
To conclude, the two descriptions are completely consistent.

Curves in the Projective Plane

In the affine plane, $\mathbb{A}^2$, an *algebraic curve* is the set of solution to a polynomial in two variables: $f(x,y) = 0$

In the projective plane we have triples as coordinates so in general we need three variables. $\mathbb{P}^2$ consists of homogeneous coordinates so it would make sense to study polynomials $F(X,Y,Z) = 0$ that has the property

$$F(a,b,c) = 0 \ \Rightarrow \ \ F(ta,tb,tc) = 0 \text{ for some non-zero } t$$

Such polynomials are called *homogeneous polynomials.* Homogeneous polynomials of *degree* d satisfy the following condition:

$$F(tX,tY,tZ) = t^d F(X,Y,Z)$$

A *projective curve* C in $\mathbb{P}^2$ is the set of solutions to the equation

$$C \ : \ F(X,Y,Z) = 0$$

The set of *rational points* on C, $C(\mathbb{Q})$, is defined by

$$C(\mathbb{Q}) = \{[a,b,c] \in \mathbb{P}^2 \ ; \ F(a,b,c) = 0 \text{ and } a,b,c \text{ are rational numbers}\}$$

An immediate observation is that notions of rational points and integer points coincide on the projective place because integers are multiples of rationals and multiples of homogeneous coordinates are invariant.

Intersections of Projective Curves

In general, two curves C_1, C_2 of degrees d_1 and d_2 will have an expected $d_1 d_2$ intersections. It is important to note that these intersections may have a *multiplicity* to them just as roots to a polynomial may have multiplicities, afterall, the intersections *are* roots to polynomials.

Now we need to bring along *tangents* to curves. From differential calculus, the tangent line to an affine curve C at some point $P = (r,s)$ is given by

$$\frac{\partial f}{\partial x}(r,s)(x-r)+\frac{\partial f}{\partial y}(r,s)(y-s)=0$$

What happens if both partial derivatives vanish when evaluated at (r,s)? The tangent will not be defined at point P and we call such a point a *singular point* of the curve C.
So P is a singular point if

$$\frac{\partial f}{\partial x}(P)=\frac{\partial f}{\partial y}(P)=0$$

If C does not contain any singular points, we say that C is a *non-singular curve* or C is a *smooth curve.*

The tangents of curves are closely related to the number of intersections between curves. Here are two examples to demonstrate the use of tangents:

Example 1

Consider intersection points of the line C_1 and the conic C_2 given by

$$C_1 : x+1=0 \qquad \text{and} \qquad C_2 : y-x^2=0$$

It looks like there is only one intersection at $(-1,1)$, but that is only the affine plane. These are curves of degrees one and two so in the projective plane we expect to have 2 intersections. The algebraic approach would be to homogenise the equations, giving

$$C_1' : X+Z=0 \qquad \text{and} \qquad C_2' : YZ-X^2=0 \qquad \Rightarrow \qquad YZ-Z^2=0$$

We can see the solutions lie at $Z=0,-Y=-X$, we take the two solutions as $[0,1,0]$ and $[-1,1,1]$. It is natural to ask the question why we take $Y=1$ even though it is free variable, the answer is that it doesn't matter what y equals to because we are using homogeneous coordinates. Any Y would give the same solutions for coordinates in the form $[0,Y,0]$ and $[-Y,Y,Y]$.
Now that things look well algebraically, we have a rather neat(er) answer geometrically. The key point is to think about the tangent of C_2 as x approaches infinity; C_2 tends to being vertical. Since at infinity C_1 and C_2 will have the same direction, they simply intersect for the second time at infinity.

Example 2

We look at another case where the tangents gives us extra information.
The line $C_1 : x + y - 2 = 0$ and the conic $C_2 : x^2 + y^2 - 2 = 0$ seem to only intersect at the point $(1, 1)$. Even if we homogenise the equations we only obtain the point $[1, 1, 1]$. This is because if we substitute equation for C_1 into C_2, we find $2(y-1)^2 = 0$, so in fact we have a root of multiplicity two and we count the intersection twice. At the intersection, we see that the tangents for the curves coincide. This situation is similar to a parabola intersecting the x axis at the same point *twice*, and the tangent of the parabola at the intersection is parallel to the x axis.

Note: Intersections can also have multiplicities at singular points even if the tangents are not parallel at the intersection.

It is possible for two curves to intersect *infinitely* many time. This happens when every point of a curve lies inside another curve.
For a general curve C given by $C : f(x, y) = 0$, we can factor f into irreducible polynomials

$$f(x, y) = p_1(x, y)p_2(x, y) \dots p_n(x, y)$$

This factorisation is unique since $\mathbb{C}(x, y)$ is a unique factorisation domain. Each of the polynomials p_i form a curve $p_i = 0$, which are called *components* of C. C is then the union of all of it's components and so each component lies completely inside C.
We say that C is *irreducible* if it contains one component only. If C_1 and C_2 have *no common components* then each of their components are distinct and C_1 and C_2 have finitely many intersections.

For two curves C_1 and C_2 with no common components, assign a *multiplicity* or *intersection index* $I(C_1 \cap C_2, P)$ for some point $P \in \mathbb{P}^2$.
The intersection index has the following properties:

- If $P \notin C_1 \cap C_2$, then $I(C_1 \cap C_2, P) = 0$
- If $P \in C_1 \cap C_2$, P is not a singular point of C_1 and C_2 and if C_1 and C_2 have different tangents at P then $I(C_1 \cap C_2, P) = 1$ (in this case we say C_1 and C_2 intersect *transversally* at P)
- If $P \in C_1 \cap C_2$ and if C_1 and C_2 do not intersect transversally at P, then $I(C_1 \cap C_2, P) \geq 2$

We are now ready to state our first theorem.

Bezout's Theorem
Let C_1 and C_2 be projective curves with no common components, then

$$\sum_{P \in C_1 \cap C_2} I(C_1 \cap C_2, P) = (deg C_1)(deg C_2)$$

If all intersections are transversal, then

$$\# C_1 \cap C_2 = (deg C_1)(deg C_2)$$

and in any case, the following holds

$$\#C_1 \cap C_2 \leq (degC_1)(degC_2)$$

Bezout's theorem implies that if two conics, C_1 and C_2, contains 5 common points, they either contain a common component, or $C_1 = C_2$.
To generalise, if two curves C_1 and C_2 with no common components both of degree d are to go through $d^2 + 1$ prescibed points on the plane, then $C_1 = C_2$.

Now move on and study a case involving cubic curves. Suppose that C_1 and C_2 are cubic curves intersecting at 9 distinct points $P_1, \ldots, P_9$. Further suppose that another cubic curve D goes through the first 8 points $P_1, \ldots, P_8$.
We claim that D must also go through the point P_9.

Denote the collection of cubic curves in $\mathbb{P}^2$ by $\mathcal{C}^{(3)}$. If $C \in \mathcal{C}^{(3)}$, then C is given by the homogeneous equation

$$C : aX^3 + bX^2Y + cXY^2 + dY^3 + eX^2Z + fXZ^2 + gY^2Z + hYZ^2 + iZ^3 + jXYZ = 0$$

So C is determined by ten coefficients [a,b,...,j] and we could see that $\mathcal{C}^{(3)}$ is isomorphic to $\mathbb{P}^9$. If $P_1, ...P_n \in \mathbb{P}^2$ then there is a one-to-one correspondence between

$$\{C \in \mathcal{C}^{(3)}; P_1, ..., P_n \in C\} \text{ and } \{\text{simultaneous solutions to } n \text{ homogeneous equations in } \mathbb{P}^9\}$$

This means if we fix 8 points on the plane, we will be left with a space of dimension 2. So any 2 linearly independent solutions in this space will span the whole space. But we already have 2 such solutions, $\mathbf{v}_1$ and $\mathbf{v}_2$ $C_1 : F_1(X, Y, Z)$ and $C_2 : F_(X, Y, Z)$. This means that all solutions inside this space must be in the form $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$, in particular D will be. So D is given by $D : \lambda_1 F_1(X, Y, Z) + \lambda_2 F_2(X, Y, Z) = 0$. Since $F_1(P_9) = F_1(P_9) = 0$, P_9 also lies in D.

$\square$

This is a special case of the *Cayley-Bacharach Theorem*, the case of curves of degree 3. This could be used to prove *Pascal's Theorem.* Here, I will give a brief set up and proof for Pascal's Theorem.

Let C be a smooth conic and choose *any* six points lying on the conic. Label the six points $P_1, P_2, ..., P_6$ and form a hexagon by join a line from P_1 to P_2, P_2 to P_3, etc. Now extend a line from each of the sides of the hexagon so that we obtain 3 intersection points from the extension lines.
Without loss of generality,

$$\overline{P_1P_2} \cap \overline{P_4P_5} = \{Q_1\}, \qquad \overline{P_2P_3} \cap \overline{P_5P_6} = \{Q_2\}, \qquad \overline{P_3P_4} \cap \overline{P_6P_1} = \{Q_3\}$$

where $\overline{P_iP_j}$ denotes the extended line of the side P_iP_j.

Pascal's Theorem

The three points Q_1, Q_2, Q_3 constructed above lie on a line.

Proof

Consider two cubic curves given by

$$C_1 : \overline{P_1P_2} \cup \overline{P_3P_4} \cup \overline{P_5P_6} \qquad C_2 : \overline{P_2P_3} \cup \overline{P_4P_5} \cup \overline{P_6P_1}$$

Observe that all of the nine points $P_1, P_2, P_3, P_4, P_5, P_6, Q_1, Q_2, Q_3$ lie on both C_1 anc C_2.
Now let D be the cubic curve given by

$$D = C \cup \overline{Q_1Q_2}$$

D clearly contains the eight points $P_1, P_2, P_3, P_4, P_5, P_6, Q_1, Q_2$. By the Cayley-Bacharach Theorem, Q_3 must also line on D, hence Q_3 lies on either C or $\overline{Q_1Q_2}$. If Q_3 lies on C the line $\overline{P_6P_1}$ will intersect C at three points, namely P_6, P_1, Q_3, contradicting Bezout's Theorem.
Hence Q_3 lies on the line $\overline{Q_1Q_2}$.
$\square$

This beautiful proof uses Cayley-Bacharach Theorem for cubic curves, but one astonishing fact is that the geometric construction of theorem did *not* involve any cubics.

Exercise

1. Let P_1, P_2, P_3, P_4, P_5 be five distinct points in $\mathbb{P}^2$.
   (a) Show that there exists a conic C (i.e. a curve of degree two) passing through the five points.
   (b) Show that C is unique if and only if no four of the five points lie on a line.
   (c) Show that C is irreducible if and only if no three of the five points lie on a line.

Solution

1. (a) The general conic in $\mathbb{P}^2$ has the form

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$$

The 6-tuple of letters a, b,..., f determines the conics. This 6-tuple corresponds to the projective space $\mathbb{P}^5$ which contains 6 variables. If 5 points were prescribed on $\mathbb{P}^2$, there would be an equivalent 5 linear conditions on 6 variables in $\mathbb{P}^5$. This means we will be left with a space of solutions of one dimension and hence there exists a conic which goes through the points P_1, P_2, P_3, P_4, P_5.

(b) $(\Rightarrow)$
Assume that C is unique and suppose that four of the points lie on the same line. WLOG, let these four points be P_1, P_2, P_3, P_4. Let $\overline{P_iP_j}$ denote the line defined by the two points P_i and P_j. Then the conic $C = \overline{P_1P_4} \cup \overline{P_5P_i}$, for each $1 \leq i \leq 4$, clearly contains all 5 points and so is contradicting the uniqueness of C. So four of the points cannot lie on the same line.
$(\Leftarrow)$
Assume that no four of the five points lie on a line. In the case where there are three points on a line, call this line l_1. The remaining two points define a unique line, l_2. This gives a unique C given by $C = l_1 \cup l_2$. So suppose there are at most two points on the same line, then we can use Bezout's theorem to show that the conic C is unique.

(c) $(\Rightarrow)$
Assume C is irreducible so that C is given by a $F(X, Y, Z) = 0$ where F is irreducible. Suppose there exists three points on the same line. Then following from part (b), C is a union of two lines l_1 and l_2 given by $l_1 : f_1(X, Y, Z) = 0$ and $l_2 : f_2(X, Y, Z) = 0$. We can then write

$$C : F(X, Y, Z) = f_1(X, Y, Z)f_2(X, Y, Z) = 0$$

contradicting that F is irreducible. So no three points can lie on the same line.
$(\Leftarrow)$
Assume that no three points lie on the same line so that at least three lines will be required to enclose all five points. Suppose that F is reducible and so C is reducible. F is a polynomial of degree 2 so if it is reducible, F will be written as $F = p_1p_2$ where p_1 and p_2 are polynomials of degree 1. This means that C is the union of *two* lines going though 5 points. Contradiction. Hence C is irreducible.