Ideal, Varieties, and Algorithms: Chapter 1 Exercises

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June 3, 2008

§2, Question 6:

a. Prove that a single point $(a_1, ..., a_n) \in k^n$ is an affine variety.
b. Prove that every finite subset of k^n is an affine variety. Hint:Lemma 2 will be useful.
Solutions:
a. Consider the set $V = \{(a_1, ..., a_n)\}$
Then for V to be an affine variety, there must a collection of polynomials $f_1, ..., f_n \in k[x_1, ..., x_n]$ such that f vanishes at $(a_1, ..., a_n)$.
Let

$$f_i := (x_i - a_i)$$

So we found a set of polynomials that vanish at V
b. Lemma 2 States: If V and W are Affine varieties, then so are $W \cap V$ and $W \cup V$
Also in one of the exercices we proved that a finite unions and intersection of affine varieties are also varieties.
Let $W \subset k^n$ where W is finite. Also let $V_i := (a_{i1}, ..., a_{in})$ where V_i are all subsets of W consisting of exactly one point.
Then $W = \bigcup V_i \Rightarrow$ by Lemma 2 that W is also an affine variety.

§3, question 8

Consider the curve $y^2 = cx^2 - x^3$ where $c > 0$
a. Show at the line will meet at 0,1,2, or 3 points.
b. Show that a nonvertical line through the origin $(y = mx)$ meets the curve at one other point when $m^2 \neq c$
c. Now draw the vertical line $x = 1$. Given a point $(1, t)$ on this line, drawthe line connecting $(1, t)$ to the origin. This will intersect the curve in a point (x, y). Draw a picture to illustrate this, and argue geometrically that this gives a parametrization of the entire curve.
d. Show that the geometric description from part c leads to the parametrization

$$x = c - t^2, y = t(c - t^2)$$

Solutions: a. if $x = a$ then $y^2 = ca^2 - a^3 = a^2(c - a)$
if $a > c$ then right-hand side of the equation is negative, so the line does not meet the curve.
if $a = c$ it meets the line at $(0, 0)$.
if $a < c$ it meets the line at 2 points

if the equation of the line is $y = mx + b$, then

$$(mx + b)^2 = cx^2 - x^3$$

which simplifies into

$$x^3 + x^2(m^2 - c) + 2mbx + b^2 = 0$$

so this equation has at most 3 roots, which implies it meets the curve at most 3 points

b. take $b = 0$ we get

$$x(x^2 + x(m^2 - c)) = 0$$

so the roots to this equation is 0 or $2(m^2 - c)$

c. as t ranges from $-\infty$ to ∞ the line meets the curve at all points.

d. Consider the colinear points (x, y), $(0, 0)$, $(1, t)$
by calculating the slope of this line we get $y = tx$, plugin it in we get

$$t^2x^2 = cx^2 - x^3$$

canceling out x^2, we get

$$x = c - t^2$$

but $y = tx$, so

$$y = t(c - t^2)$$

§4, Question 8

The ideal $I(V)$ of a variety has a special property not shared by all ideals. Specifically, we define an ideal I to be radical if whenever a power f^m of a polynomial f is in I , then $f \in I$. More succinctly, I is radical when $f \in I$ if and only if $f^m \in I$ for some positive integer m.
a. Prove that $I(V)$ is always a radical ideal.
b. Prove that $\langle x^2, y^2 \rangle$ is not a radical ideal. This implies that $\langle x^2, y^2 \rangle \neq I(V)$

for any variety $V \subset k^2$.

Solutions: a. Clearly if $f \in I(V) \Rightarrow f^m \in I(V)$
Now, let $f^m \in I(V)$ then $f = 0$ for all elements of V,
but $f^m(a_1, ..., a_n) = (f(a_1, ..., a_n))^m$ which implies $f \in I(V)$

b. $x^2 \in \langle x^2, y^2 \rangle$ but $x \notin \langle x^2, y^2 \rangle$

§5, question 11

In this exercise we will study the one-variable case of the consistency problem from §2. Given $f_1, ..., f_s \in k[x]$, this asks if there is an algorithm to decide whether $V(f_1, ..., f_s)$ is nonempty.
a. Let $f \in \mathbb{C}[x]$ be a nonzero polynomial. Then use the Fundamental theorem of Algebra to show that $V(f) = \emptyset$ if and only if f is constant.
b. If $f_1, ..., f_s \in [x]$, prove $V(f_1, ..., f_s) = \emptyset$ if and only if $GCD(f_1, ..., f_s) = 1$.

Solutions: a. if f is a non-zero constant, then it never vanishes , so $V(f) = \emptyset$
Now suppose $V(f) = \emptyset$, by the fundamental theorem of algebra if f was non-constant polynomial, $V(f) \neq \emptyset$, so f must be a constant polynomial.

b. Suppose $GCD(f_1, ..., f_s) = 1$
Its was proved in a previous exercise by using the Fundamental Theorem of Algebra that any f_i,$f_j \in V(f_1, ..., f_s) \subset \mathbb{C}[x]$ can be written as

$$f_i = c(x - a_1)(x - a_2)...(x - a_n)$$

$$f_j = d(x - b_1)(x - b_2)...(x - b_n)$$

where c,$d \in \mathbb{C}[x]$
By our assumption f_i has no common factor other than 1 with f_j for $i \neq j$ which means that for all $(x - a_q)$ that divide f_i where $1 \leq q \leq n$ then $(x_q - a_q)$ cannot divide f_j
$f_i(a_1) = 0$ but $f_j(a_1) \neq 0$ for $i \neq j$
which implies that $V(f_1, ..., f_s) = \emptyset$

Now suppose $V(f_1, ..., f_s) = \emptyset$ but $GCD(f_1, ..., f_s) \neq 1$
Say $GCD(f_1, ..., f_s) = (x - a_q)h$ where $1 \leq q \leq n$ and $h \in \mathbb{C}[x]$
so f_i can be written as

$$f_i = (x - a_q)hg$$

where $g \in \mathbb{C}[x]$
then $f_i(a_q) = 0$ for all i, which implies $a_q \in V(f_i, ..., f_s) \neq \emptyset$ which is a contradiction.