2201 Sheet 4

Minhyong Kim (following Richard Hill)

December 12, 2008

Questions 2, 3 and 4 are assessed. The other questions are optional. The assessed homework is to be handed in on Thursday 30th October at the lecture.

If you find a mistake in any question then please email me (`minhyong.kim@ucl.ac.uk`).

1. For each of the following matrices A, calculate $\mathrm{rank}(A)$ and $\mathrm{null}(A)$ and find a basis for $\ker(A)$.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 4 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 4 & 4 & 8 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Calculate $\mathrm{ch}_A(x), m_A(x), \mathrm{ch}_B(x), m_B(x)$.

   Diagonalize A and B (i.e. find P, Q such that $P^{-1}AP$ and $Q^{-1}BQ$ are diagonal.)

3. Let $V = \mathbb{C}^4$ and consider the linear map $V \to V$ given by the matrix:

$$A = \begin{pmatrix} 12 & -6 & 6 & -6 \\ 2 & 21 & 21 & 51 \\ -3 & 12 & 12 & 30 \\ 1 & -9 & -9 & -21 \end{pmatrix}.$$

   Given that $\mathrm{ch}_A(X) = (X-6)^4$, calculate

   (i) The power b such that $m_A(X) = (X-6)^b$;

   (ii) A basis for each generalized eigenspace $V_t(6)$ $(t = 1, \ldots, b)$.

4. Let V be the vector space of polynomials over $\mathbb{C}$ of degree ≤ 10 and let $D : V \to V$ be the linear map defined by

$$D(f) = \frac{\mathrm{d}f}{\mathrm{d}x}.$$

   (i) Show that $D^{11} = 0$.

   (ii) Deduce that 0 is the only eigenvalue of D.

   (iii) Find a basis for the generalized eigenspaces $V_1(0)$, $V_2(0)$ and $V_3(0)$.

5. Prove that if $T : V \to V$ is linear and $f \in k[X]$ is an irreducible, monic polynomial satisfying $f(T) = 0$ then $f = m_T$.

Cayley–Hamilton Theorem *For any* $T \in \mathrm{End}(V)$ *we have* $\mathrm{ch}_T(T) = 0$.

Definition Let V be a finite dimensional vector space over a field k and $T : V \to V$ a linear map. A *minimal polynomial* of T is a monic polynomial $m \in k[X]$ such that

- $m(T) = 0$;
- if $f(T) = 0$ and $f \neq 0$ then $\deg f \geq \deg m$.

Theorem *Every linear map* $T : V \to V$ *has a unique minimal polynomial* m_T.

Theorem $f(T) = 0$ *if and only if* $m_T | f$.

Corollary *If* $T : V \to V$ *is a linear map then* $m_T | \mathrm{ch}_T$.

Proposition *Let* v *be an eigenvector of* T *with eigenvalue* $\lambda \in k$. *Then for any polynomial* $f \in k[X]$,

$$(f(T))(v) = f(\lambda) \cdot v.$$

Theorem *If* $T : V \to V$ *is linear and* $\lambda \in k$ *then the following are equivalent:*

(i) λ *is an eigenvalue of* T.

(ii) $m_T(\lambda) = 0$.

(iii) $\mathrm{ch}_T(\lambda) = 0$.

Definition Let V be a finite dimensional vector space over a field k, and let $\lambda \in k$ be an eigenvalue of a linear map $T : V \to V$. We define for $t \in \mathbb{N}$ the t-th generalized eigenspace by:

$$V_t(\lambda) = \ker((\lambda \cdot \mathrm{id} - T)^t).$$

Note that $V_1(\lambda)$ is the usual eigenspace (i.e. the set of eigenvectors together with zero).

Lemma *If* $f, g \in k[x]$ *satisfy* $\mathrm{hcf}(f, g) = 1$ *and* T *is as above then*

$$\ker(fg(T)) = \ker(f(T)) \oplus \ker(g(T)).$$

Primary Decomposition Theorem *If* V *is a finite dimensional vector space and* $T : V \to V$ *is linear, with distinct eigenvalues* $\lambda_1, \dots, \lambda_r \in k$, *minimal polynomial*

$$m_T(X) = \prod_{i=1}^{r} (X - \lambda_i)^{b_i},$$

then

$$V = V_{b_1}(\lambda_1) \oplus \cdots \oplus V_{b_r}(\lambda_r).$$

Sheet 4 Solutions
Richard Hill

2. Since these matrices are both upper triangular we can write down their characteristic polynomials:

$$\mathrm{ch}_A(X) = (X-1)^2(X-2), \quad \mathrm{ch}_B(X) = (x-1)(x-2)^2.$$

Then $(A-I)(A-2I) = 0 = (B-I)(B-2I)$ so $m_A(x) = m_B(x) = (X-1)(X-2)$.
A has eigenvalues 1 and 2. For the eigenvalue 1 we have two linearly independent eigenvectors $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $v_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. For the eigenvalue 2 we have an eigenvector $v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ So we obtain

$$P = \begin{pmatrix}1&0&1\\0&0&1\\0&1&0\end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix}1&&\\&1&\\&&2\end{pmatrix}.$$

For B we also have eigenvalues 1 and 2. For the eigenvalue 1 we have an eigenvector $v_1 = \begin{pmatrix}0\\-2\\1\end{pmatrix}$.

For the eigenvalue 2 we have two linearly independent eigenvectors $v_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

So we obtain

$$Q = \begin{pmatrix}0&1&0\\-2&0&1\\1&0&0\end{pmatrix}, \quad Q^{-1}BQ = \begin{pmatrix}1&&\\&2&\\&&2\end{pmatrix}.$$

3. (i) We have

$$\begin{aligned}
(A-6I_3) &= \begin{pmatrix}6&-6&6&-6\\2&15&21&51\\-3&12&6&30\\1&-9&-9&-15\end{pmatrix} \neq 0,\\
(A-6I_3)^2 &= \begin{pmatrix}0&0&0&0\\30&6&-6&6\\18&0&0&0\\-12&-6&6&-6\end{pmatrix} = 6\begin{pmatrix}0&0&0&0\\5&1&-1&1\\3&0&0&0\\-2&-1&1&-1\end{pmatrix} \neq 0,\\
(A-6I_3)^3 &= 6\begin{pmatrix}0&0&0&0\\36&-36&36&-36\\18&-18&18&-18\\-18&18&-18&18\end{pmatrix} = 108\begin{pmatrix}0&0&0&0\\2&-2&2&-2\\1&-1&1&-1\\-1&1&-1&1\end{pmatrix} \neq 0,\\
(A-6I_3)^4 &= 0.
\end{aligned}$$

Therefore $m_A(X) = (X-6)^4$.

(ii) We have (by row reduction):

$$V_1(6) = \ker\begin{pmatrix}6&-6&6&-6\\2&15&21&51\\-3&12&6&30\\1&-9&-9&-15\end{pmatrix} = \ker\begin{pmatrix}1&0&0&0\\0&1&0&2\\0&0&1&1\end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix}0\\-2\\-1\\1\end{pmatrix}\right\},$$

$$V_2(6) = \ker\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 1 & -1 & 1 \\ 3 & 0 & 0 & 0 \\ -2 & -1 & 1 & -1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$V_3(6) = \ker\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -2 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$V_4(6) = \ker(0) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

4. (i) Differentiating a polynomial reduces the degree, so if we differentiate an element of V eleven times we have 0. Therefore $D^{11} = 0$.

(ii) By one of the theorems on the sheet, m_D is a factor of X^{11}. Hence $m_D(X) = X^b$ for some $b \leq 11$. It follows that 0 is the only zero of m_D. Hence by another theorem on the sheet, 0 is the only eigenvalue.

(iii)

$$\begin{aligned} V_1(0) &= \left\{ f \in V : \frac{\mathrm{d}f}{\mathrm{d}x} = 0 \right\} = \operatorname{span}\{1\}, \\ V_2(0) &= \left\{ f \in V : \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0 \right\} = \operatorname{span}\{1, x\}, \\ V_3(0) &= \left\{ f \in V : \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = 0 \right\} = \operatorname{span}\{1, x, x^2\}. \end{aligned}$$