6.3 Orthogonal and orthonormal vectors

Definition. *We say that 2 vectors are orthogonal if they are perpendicular to each other. i.e. the dot product of the two vectors is zero.*

Definition. *We say that a set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ are mutually orthogonal if every pair of vectors is orthogonal. i.e.*

$$\vec{v}_i.\vec{v}_j = 0, \text{ for all } i \neq j.$$

Example. The set of vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ is mutually orthogonal.

$$\begin{aligned} (1,0,-1).(1,\sqrt{2},1) &= 0 \\ (1,0,-1).(1,-\sqrt{2},1) &= 0 \\ (1,\sqrt{2},1).(1,-\sqrt{2},1) &= 0 \end{aligned}$$

Definition. *A set of vectors S is orthonormal if every vector in S has magnitude 1 and the set of vectors are mutually orthogonal.*

Example. We just checked that the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

are mutually orthogonal. The vectors however are not normalized (this term is sometimes used to say that the vectors are not of magnitude 1). Let

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \\ \vec{u}_2 &= \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix} \\ \vec{u}_3 &= \frac{\vec{v}_3}{|\vec{v}_3|} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{pmatrix} \end{aligned}$$

The set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal.

Proposition *An orthogonal set of non-zero vectors is linearly independent.*

6.4 Gram-Schmidt Process

Given a set of linearly independent vectors, it is often useful to convert them into an orthonormal set of vectors. We first define the projection operator.

DEFINITION. *Let $\vec{u}$ and $\vec{v}$ be two vectors. The projection of the vector $\vec{v}$ on $\vec{u}$ is defined as folows:*

$$Proj_{\vec{u}}\vec{v} = \frac{(\vec{v}.\vec{u})}{|\vec{u}|^2}\vec{u}.$$

EXAMPLE. Consider the two vectors $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
These two vectors are linearly independent.
However they are not orthogonal to each other. We create an orthogonal vector in the following manner:

$$\begin{aligned} \vec{v}_1 &= \vec{v} - (\text{Proj}_{\vec{u}}\vec{v}) \\ \text{Proj}_{\vec{u}}\vec{v} &= \frac{(1)(1)+(1)(0)}{(\sqrt{1^2+0^2})^2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \vec{v}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$\vec{v}_1$ thus constructed is orthogonal to $\vec{u}$.

The Gram-Schmidt Algorithm:

Let $v_1, v_2, ..., v_n$ be a set of n linearly independent vectors in $\mathcal{R}^n$. Then we can construct an orthonormal set of vectors as follows:

Step 1. Let $\vec{u}_1 = \vec{v}_1$. $\vec{e}_1 = \frac{\vec{u}_1}{|\vec{u}_1|}$.

Step 2. Let $\vec{u}_2 = \vec{v}_2 - \text{Proj}_{\vec{u}_1}\vec{v}_2$. $\vec{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|}$.

Step 3. Let $\vec{u}_3 = \vec{v}_3 - \text{Proj}_{\vec{u}_1}\vec{v}_3 - \text{Proj}_{\vec{u}_2}\vec{v}_3$. $\vec{e}_3 = \frac{\vec{u}_3}{|\vec{u}_3|}$.

Step 4. Let $\vec{u}_4 = \vec{v}_4 - \text{Proj}_{\vec{u}_1}\vec{v}_4 - \text{Proj}_{\vec{u}_2}\vec{v}_4 - \text{Proj}_{\vec{u}_3}\vec{v}_4$. $\vec{e}_4 = \frac{\vec{u}_4}{|\vec{u}_4|}$.

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Example. We will apply the Gram-Schmidt algorithm to orthonormalize the set of vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

To apply the Gram-Schmidt, we first need to check that the set of vectors are linearly independent.

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(0-1) - 1((-1)(2) - (1)(1)) + 1((-1)(1) - 0) = 1 \neq 0.$$

Therefore the vectors are linearly independent.
Gram-Schmidt algorithm:

Step 1. Let

$$\begin{aligned} \vec{u}_1 &= \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \vec{e}_1 &= \frac{\vec{u}_1}{|\vec{u}_1|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

Step 2. Let

$$\begin{aligned} \vec{u}_2 &= \vec{v}_2 - \text{Proj}_{\vec{u}_1} \vec{v}_2 \\ \text{Proj}_{\vec{u}_1} \vec{v}_2 &= \frac{(1,0,1).(1,-1,1)}{1^2 + (-1)^2 + 1^2} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \vec{u}_2 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} \\ \vec{e}_2 &= \frac{\vec{u}_2}{|\vec{u}_2|} = \frac{3}{\sqrt{6}} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}. \end{aligned}$$

Step 3. Let

$$\begin{aligned}
\vec{u}_3 &= \vec{v}_3 - \mathrm{Proj}_{\vec{u}_1}\vec{v}_3 - \mathrm{Proj}_{\vec{u}_2}\vec{v}_3 \\
\mathrm{Proj}_{\vec{u}_1}\vec{v}_3 &= \frac{(1,1,2).(1,-1,1)}{1^2+(-1)^2+1^1}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
\mathrm{Proj}_{\vec{u}_2}\vec{v}_3 &= \frac{(1,1,2).(1/3,2/3,1/3)}{(1/3)^2+(2/3)^2+(1/3)^2}\begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} \\
\vec{u}_3 &= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} \\
\vec{e}_3 &= \frac{\vec{u}_3}{|\vec{u}_3|} = \sqrt{2}\begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}.
\end{aligned}$$

Example. Consider the vectors $\{[3,0,4],[-1,0,7],[2,9,11]\}$ Check that the vectors are linearly independent and use the Gram-Schmidt process to find orthogonal vectors.

Ans. $\{[3,0,4],[-4,0,3],[0,9,0]\}$ Check that the vectors are mutually orthogonal.