Topological Invariants in braid theory

Mitchell A Berger
Mathematics, University College London
Gower Street London, WC1E 6BT, UK
m.berger@ucl.ac.uk

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Abstract

Many invariants of knots and links have their counterparts in braid theory. Often these invariants are most easily calculated using braids. A braid is a set of n strings stretching between two parallel planes. This review demonstrates how integrals over the braid path can yield topological invariants. The simplest such invariant is winding number – the net number of times two strings in a braid wrap about each other. But other, higher order invariants exist. The mathematical literature on these invariants usually employs techniques from algebraic topology that may be unfamiliar to physicists and mathematicians in other disciplines. The primary goal of this paper is to introduce higher order invariants using only elementary differential geometry.

Some of the higher order quantities can be found directly by searching for closed one-forms. However, the Kontsevich integral provides a more general route. This integral gives a formal sum of all finite order topological invariants. We describe the Kontsevich integral, and prove that it is invariant to deformations of the braid.

Some of the higher order invariants can be used to generate Hamiltonian dynamics of n particles in the plane.The invariants are expressed as complex numbers; but only the real part gives interesting topological information. Rather than ignoring the imaginary part, we can use it as a Hamiltonian. For $n = 2$, this will be the Hamiltonian for point vortex motion in the plane. The Hamiltonian for $n = 3$ generates more complicated motions.

Braid theory simplifies many aspects of the theory of knots. Braids can be readily classified using the braid group [8]. Also, every braid can be closed into a knot by joining pairs of endpoints from bottom and top planes. Suppose, for example, the beginning and endpoints of each of the n strings have the same horizontal positions. Then we can form an n-component link from the braid by identifying top and bottom planes. The winding numbers between two strings become Gauss linking numbers. For some links such as the Borromean rings [13], the Gauss linking numbers vanish, and yet the strings cannot be pulled apart (see figure 1). For these links, we must turn to higher order linking numbers, discovered in the 1950s by Massey and Milnor (see [16] for a review). These higher order quantities can be expressed as integrals over vector fields [19, 20, 4, 15, 22, 18]. The higher order winding numbers for braids [5, 6] described in §1 correspond to these linking numbers.

Vassiliev [23] considered the topology of singular knots. A singular knot has a finite number of double points (i.e. points where a curve intersects itself). Suppose an invariant I_n has been defined for all curves with n double points. To extend the definition to curves with $n+1$ double points we imagine taking the curve at some place where two parts of the curve come close to each other, and passing those two parts through each other. At the instant of time where they pass through, there is a new double point. Before this time, the two parts may display an overcrossing, and afterwards an undercrossing (or vice versa). Calculate I_n^+ for the overcrossing, and I_n^- for the undercrossing. Then we can define $I_{n+1} \equiv I_n^+ - I_n^-$. In this way we can begin with an ordinary knot invariant I_0, and proceed to define singular knot invariants I_1, I_2, etc.

For many choices of ordinary knot invariant I_0 (e.g. linking numbers, Jones polynomials), this series will not go on forever. A finite-type or Vassiliev invariant of order m vanishes for more than m double points, i.e. $I_m \neq 0$ (for some curve), but $I_{m+1} = I_{m+2} = \ldots = 0$ (for all curves) [9]. The Gauss linking number is an order 1 invariant. In 1993 Kontsevich [17] showed that Vassiliev invariants can be found using an integral formula (for a bibliography on Vassiliev-Kontsevich theory, see [3]). Recently, Willerton [24] showed that the higher order winding number for 3-braids is a finite type (order 2) Vassiliev invariant, and showed how to obtain it from the Kontsevich integral. He also obtained expressions for similar invariants of higher order.

Section 2 looks at the Kontsevich integral and shows how it can be constructed for braids using little more than elementary differential geometry. Our discussion follows the excellent review of Vassiliev-Kontsevich knot theory by Chmutov & Duzhin [14]. As much as possible we attempt to simplify the mathematics, first by limiting the discussion to the differential geometric aspects, and second by deriving the integral for braids rather than knots. We thus ignore the Vassiliev theory of singular knots.

Section 3 employs a second order invariant as a Hamiltonian for particle motions in the plane [7]. This Hamiltonian is completely integrable. Numerical integration shows that the particles execute intertwining motions maximizing the growth of the corresponding invariant.

1 Higher order winding numbers

A geometrical n-braid can be described as n strings stretching between two parallel planes. Equivalently, we can think of n particles moving in a plane for an interval of time $0 \leq t \leq 1$. A space-time diagram of the motion with t in the vertical direction determines a braid. Braided space-time diagrams provide an elegant method for picturing two-dimensional dynamics [21, 12, 10]. It will be convenient to use the complex plane $\mathbb{C}$ for the horizontal surfaces. Thus the n strings or particle paths will be described by n functions $z_i(t)$, $i = 1, \ldots, n$ with $t \in [0, 1]$ (see figure 1).

The strings cannot cross through each other. Thus we can describe a geometrical braid with a single curve

$$\mathfrak{z}(t) = (z_1(t), \ldots, z_n(t)) \tag{1}$$

traveling through the configuration space $\mathfrak{C}$ (called $F_{0,n}(\mathbb{C})$ in [8])

$$\mathfrak{C} \equiv \{\mathbb{C}^n | z_i \neq z_j \,\text{for}\, i \neq j\}. \tag{2}$$

Two geometrical braids $\mathfrak{z}_0$, $\mathfrak{z}_1$ are topologically equivalent if $\mathfrak{z}_0$ can be transformed into $\mathfrak{z}_1$ by motions vanishing at the endpoints $t = 0, 1$.

To begin with, we present a few differential forms and show how they determine topological invariants.

1.1 Invariants from closed one-forms

Suppose we find a closed one-form A ($\mathrm{d}A = 0$) defined on $\mathfrak{C}$. Then by Stokes' theorem

$$\int_{\mathfrak{z}_0} A = \int_{\mathfrak{z}_1} A. \tag{3}$$

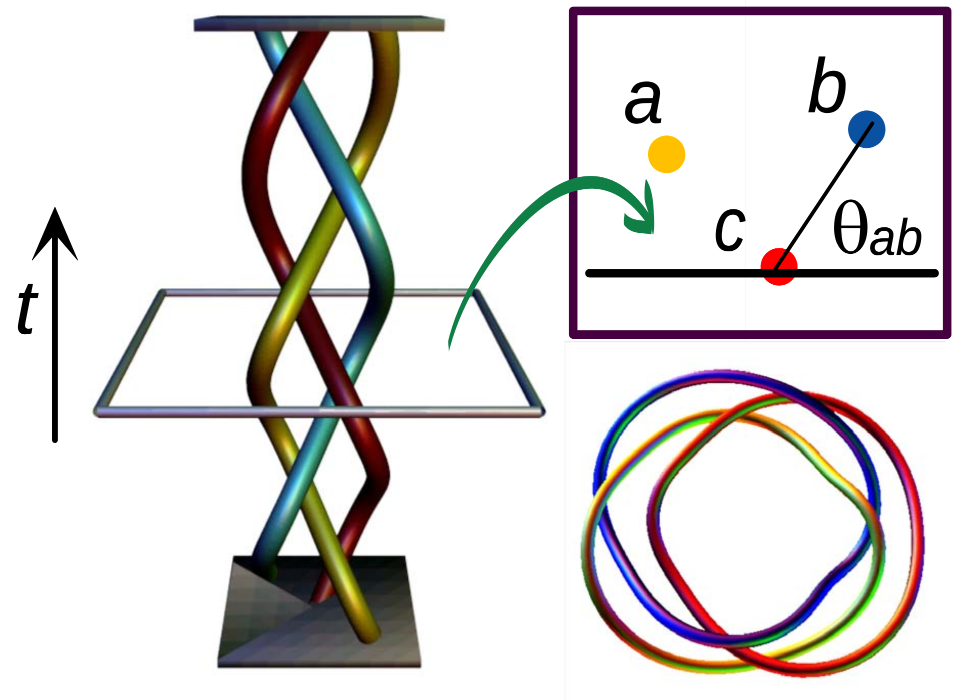


Figure 1: A geometric braid with $n = 3$ strings. This particular example is a standard pigtail braid. Also shown is the corresponding Borromean link.

Thus $\int_{\mathfrak{z}} A$ has the same value over the topological equivalence class. The most interesting topological invariants of this form are integrals of one-forms which are closed but not exact ($A \neq \mathrm{d}\psi$ for any function ψ). The exact forms merely integrate to give values at the endpoints, ignoring the structure in between.

Thus we are looking at the cohomology of braids [2]. However, there is (literally) a twist: we will especially wish to consider one-forms involving winding numbers defined on Riemann surfaces above $\mathbb{C}$. Define

$$\omega_{ij}(t) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}z_i(t) - \mathrm{d}z_j(t)}{z_i(t) - z_j(t)} \tag{4}$$

$$\lambda_{ij}(t) = \int_0^t \omega_{ij}(t')\,\mathrm{d}t'. \tag{5}$$

The one-form $\omega_{ij}(t)$ is closed. At first sight, it also seems to be exact, because $\omega_{ij} = \mathrm{d}\lambda_{ij}$. However, λ_{ij} is not single valued; in fact $2\pi\mathrm{i}\lambda_{ij}$ gives the complex logarithm of $z_i - z_j$, with imaginary part analytically continued beyond 2π. We will call $\mathrm{Re}(\lambda_{ij}) = \theta_{ij}/2\pi$ the winding number of strings i and j (see figure 1). Note that $\omega_{ji} = \omega_{ij}$.

The one-forms ω_{ij} will be used as basic building blocks for more complicated invariants. An invariant constructed from products of m of the ω forms will be called an mth order invariant. This definition is consistent with the definitions based on Vassiliev theory [14].

1.2 Cross-ratios

Some simple second order invariants arise from cross-ratio functions. Recall that given four complex numbers a, b, c, d, three cross-ratios can be defined:

$$u = \frac{(a-b)(c-d)}{(a-c)(b-d)}; \tag{6}$$

$$v = \frac{(a-c)(b-d)}{(a-d)(b-c)} = \frac{1}{1-u}; \tag{7}$$

$$w = \frac{(a-d)(b-c)}{(a-b)(c-d)} = \frac{-u}{1-u}. \tag{8}$$

Note that as v is a function of u, we can write $\mathrm{d}v = (\mathrm{d}v/\mathrm{d}u)\mathrm{d}u$, so

$$\mathrm{d}v \wedge \mathrm{d}u = 0. \tag{9}$$

Next define $\log u$ in the same way as λ_{ij}, i.e. as an analytically continued logarithm. We can now define second order invariant

$$\Theta = \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\mathfrak{z}} \log u \; \mathrm{d} \log v. \tag{10}$$

The invariance of Θ follows from the fact that $\mathrm{d}(\log v \; \mathrm{d} \log u) = (uv)^{-1} \mathrm{d}v \wedge \mathrm{d}u = 0$.

The unpleasantness of logs extending beyond $2\pi\mathrm{i}$ can be avoided by using time-ordered, or iterated integrals. The importance of iterated integrals in topology was first recognized by Chen [11]. Define

$$\Delta_2 \equiv \{(t_1, t_2) | 0 \leq t_1 \leq t_2 \leq 1\}. \tag{11}$$

Then

$$\Theta = \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\Delta_2} \mathrm{d} \log u(t_1) \wedge \mathrm{d} \log v(t_2) \tag{12}$$

$$= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_0^1 \int_0^{t_2} \frac{\mathrm{d} \log u(t_1)}{\mathrm{d}t_1} \frac{\mathrm{d} \log v(t_2)}{\mathrm{d}t_2} \; \mathrm{d}t_1 \; \mathrm{d}t_2. \tag{13}$$

This iterated form will be essential in constructing the Kontsevich integral.

Some geometric insight may be obtained by looking at associated second order invariants for three strings. If we let $d \to \infty$ in the cross-ratios, then $u \to (a-b)/(a-c)$ and $v \to (a-c)/(b-c)$. Then the integral

$$\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\mathfrak{z}} (\log v \; \mathrm{d} \log u - \log u \; \mathrm{d} \log v) \tag{14}$$

becomes

$$2\Psi_{abc} = \int (\lambda_{ab} - \lambda_{bc})\omega_{ca} + (\lambda_{bc} - \lambda_{ca})\omega_{ab} + (\lambda_{ca} - \lambda_{ab})\omega_{bc}. \tag{15}$$

This invariant [5] can, for example, distinguish the pigtail braid (see figure 1), for which individual winding numbers vanish, from the trivial braid of three vertical strings. The integrand is a closed one-form by equation 9; this implies the Arnol'd identity [2]

$$\omega_{ab} \wedge \omega_{bc} + \omega_{bc} \wedge \omega_{ca} + \omega_{ca} \wedge \omega_{ab} = 0. \tag{16}$$

This identity will be needed below to prove invariance of higher order quantities.

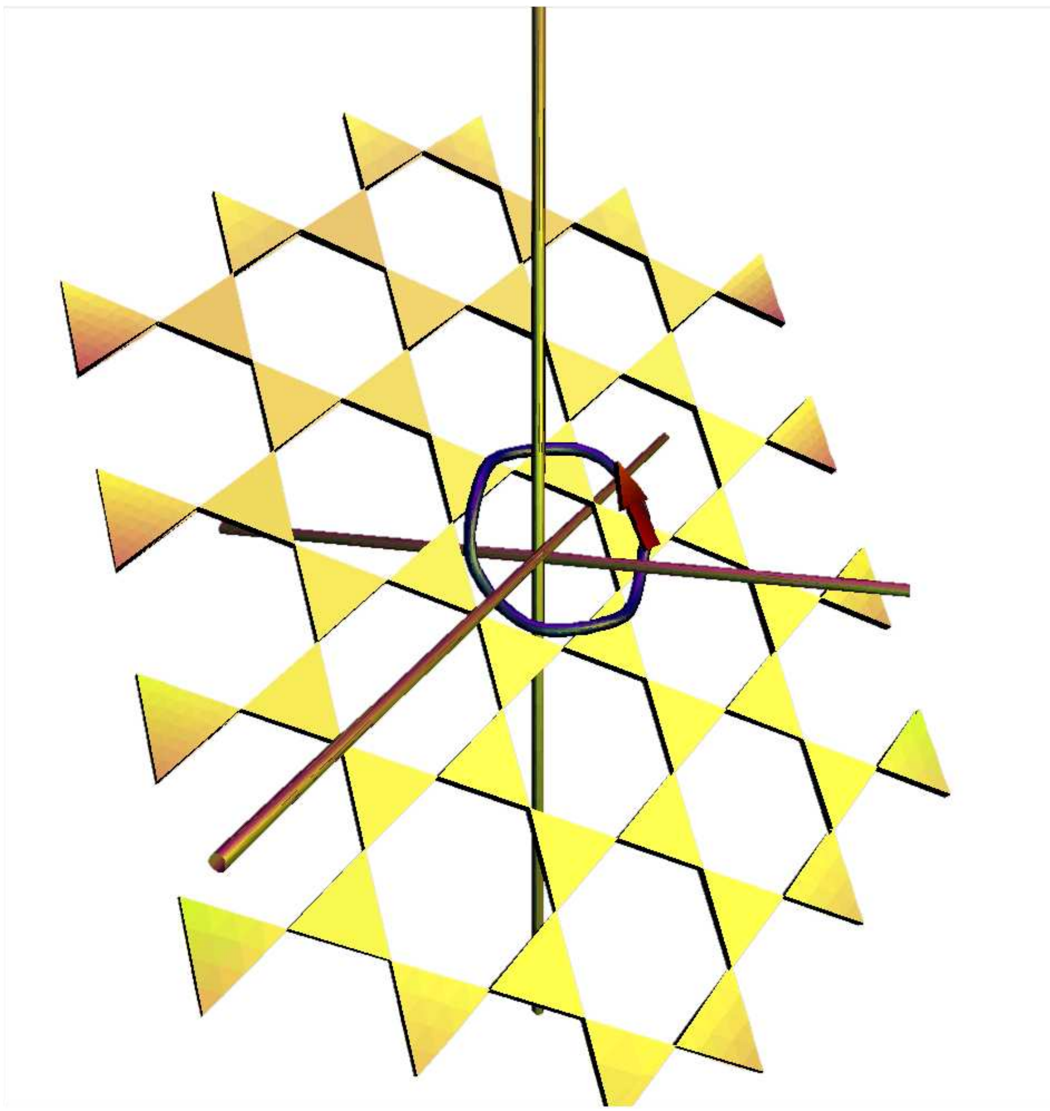


Figure 2: Space of winding numbers $\mathrm{Re}(\widetilde{\lambda}_{ab}, \widetilde{\lambda}_{bc}, \widetilde{\lambda}_{ca})$. The plane $\mathcal{P} = \mathrm{Re}(\widetilde{\lambda}_{ab} + \widetilde{\lambda}_{bc} + \widetilde{\lambda}_{ca}) = 0$ consists of accessible triangular regions and forbidden hexagonal regions. The closed path corresponds to the pigtail braid of figure 1.

Following [5, 6], we can define a phase space or winding number space consisting of the set of triplets $\mathrm{R}e(\widetilde{\lambda}_{ab}, \widetilde{\lambda}_{bc}, \widetilde{\lambda}_{ca})$ where $\widetilde{\lambda}_{ab} = \lambda_{ab} + \log(b(0) - a(0))$ (see figure 2). As the points $a(t)$, $b(t)$, and $c(t)$ move upwards in the braid, a single point $\mathrm{R}e(\widetilde{\lambda}_{ab}(t), \widetilde{\lambda}_{bc}(t), \widetilde{\lambda}_{ca}(t))$ will traverse a path in the winding number space (see figure 2). The three axes in the figure correspond to the three winding numbers; the triangular regions shown form part of a plane $\mathcal{P} = \{\mathrm{R}e(\widetilde{\lambda}_{ab} + \widetilde{\lambda}_{bc} + \widetilde{\lambda}_{ca}) = 0\}$. If a braid moves on a path perpendicular to this plane then all three angles are increasing at once, corresponding to a uniform rotation of a, b, and c.

Not all combinations of winding angles are possible. The white triangular regions correspond to allowed combinations of angles. In between are hexagonal regions; a point inside one of these regions corresponds to an impossible set of angles. At the vertices of the triangles the points a, b, and c are collinear. Also, when a, b, and c form an equilateral triangle in $\mathbb{C}$, the corresponding point in $\mathcal{P}$ is the centre of a triangular region. There are infinitely many copies of the triangular regions, corresponding to winding angles greater than 2π, i.e. different branches of the complex logarithms.

A braid corresponding to a closed path in the winding number space will encircle a certain number of the forbidden hexagonal regions. The quantity Ψ in fact equals the number of hexagons circled anticlockwise, minus the number circled clockwise [5, 6].

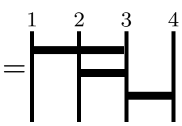
2 The Kontsevich integral

We can, perhaps, continue to find higher order invariants using special objects like the cross-ratios. For example, $\int_{\Delta_3} \mathrm{d}\log u \wedge \mathrm{d}\log v \wedge \mathrm{d}\log w$ will be a third order invariant. However, there is a general method due to Kontsevich [17] for obtaining invariants. Kontsevich wrote down an integral K, called the universal Vassiliev invariant, which is a formal sum of all finite-order invariants. To extract a particular desired invariant, the other unwanted invariants must be filtered out. This is done, as described below, by a function called a weighting system. The method works like stage lights: if green light is desired, for example, a lamp first emits white light, then a green filter removes the undesired wavelengths.

2.1 Notation

Even a simple second order invariant like Ψ_{abc} consists of several iterated integrals like $\int_{\Delta_2} \omega_{ab}(t_1) \wedge \omega_{bc}(t_2)$. Each of these integrals can be labelled by the indices of the ω forms. We can keep track of the indices in a general mth order integral by writing down an $m \times 2$ matrix $P_{\alpha\beta}$. Thus the first one-form $\omega(t_1)$ will be assigned indices P_{11} and P_{12}, the next P_{21} and P_{22}, and so on. Note that as $\omega_{ij} = \omega_{ji}$ we may assume $P_{\alpha 1} < P_{\alpha 2}$. The matrix P can also be expressed as a diagram $D(P)$. For example,

$$\omega_{13}(t_1) \wedge \omega_{23}(t_2) \wedge \omega_{34}(t_3) \Leftrightarrow P = \begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} \Leftrightarrow D(P) = \tag{17}$$



Let $\mathbb{P}_{mn}$ be the set of all possible mth order P matrices on n strings. There are $n(n-1)/2$ possible combinations for each row of P; repeats are allowed and ordering counts so $\mathbb{P}_{mn}$ has $(n(n-1)/2)^m$ elements. The Kontsevich integral will sum over all elements of $\mathbb{P}_{mn}$.

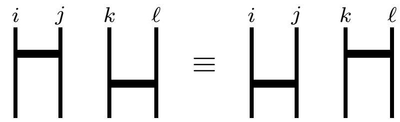
The diagram $D(P)$, known as a Gauss chord diagram, consists of n vertical lines, one for each string in the braid, and m horizontal lines representing the rows of the matrix P (we may not always draw every line or chord if we are concentrating on a few interesting ones). This object will be essential for ensuring the invariance of K to deformations of the braid. The integral K will be defined as a formal sum of Gauss diagrams, with coefficients given by the iterated integrals. However, not all diagrams may be regarded as completely independent of each other. There are two rules for combining diagrams. These rules follow from Vassiliev theory [14]; however we will merely treat them as basic assumptions which (as we will see) ensure the invariance of K.

Suppose i, j, k, and ℓ label four different strings. Also suppose P and P' differ only in the order of two neighboring rows containing these strings. In other words

$$P = \begin{pmatrix} \vdots & \vdots \\ i & j \\ k & \ell \\ \vdots & \vdots \end{pmatrix}; \quad P' = \begin{pmatrix} \vdots & \vdots \\ k & \ell \\ i & j \\ \vdots & \vdots \end{pmatrix}. \tag{18}$$

The first rule states that these two diagrams are equivalent:

$$P \equiv P'; \qquad (19)$$



The second rule involves three different strings i, j, and k. We again consider matrices P which are the same except for two neighboring rows. Let

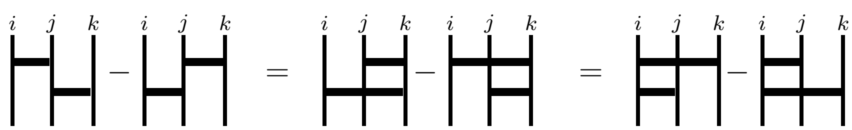
$$P_1 = \begin{pmatrix} \vdots & \vdots \\ i & j \\ j & k \\ \vdots & \vdots \end{pmatrix}; \quad P_2 = \begin{pmatrix} \vdots & \vdots \\ j & k \\ i & k \\ \vdots & \vdots \end{pmatrix}; \quad P_3 = \begin{pmatrix} \vdots & \vdots \\ i & k \\ i & j \\ \vdots & \vdots \end{pmatrix}; \qquad (20)$$

$$P_4 = \begin{pmatrix} \vdots & \vdots \\ j & k \\ i & j \\ \vdots & \vdots \end{pmatrix}; \quad P_5 = \begin{pmatrix} \vdots & \vdots \\ i & k \\ j & k \\ \vdots & \vdots \end{pmatrix}; \quad P_6 = \begin{pmatrix} \vdots & \vdots \\ i & j \\ i & k \\ \vdots & \vdots \end{pmatrix}. \qquad (21)$$

Then the following combinations will, by definition, be equivalent:

$$D(P_1) - D(P_4) \equiv D(P_2) - D(P_5) \equiv D(P_3) - D(P_6). \qquad (22)$$

In diagrams



. (23)

The Kontsevich integral can now be stated. Starting with $K_0 = 1$, $K = \sum_0^\infty K_m$ where

$$K_m = \sum_{P \,\in\, \mathbb{P}_{mn}} \int_{\Delta_m} \omega_{P_{11}P_{12}}(t_1) \wedge \cdots \wedge \omega_{P_{m1}P_{m2}}(t_m) D(P) \qquad (24)$$

$$\Delta_m = \{(t_1, \ldots, t_m) | 0 \leq t_1 \leq \ldots \leq t_m \leq 1\}. \qquad (25)$$

When evaluated on a braid, K gives a sum of diagrams with complex coefficients. A weighting system simply replaces the diagrams by numbers (e.g. 1 for the desired diagrams and 0 for the undesired diagrams). The weighting system must be consistent with the two equivalence rules for the diagrams.

2.2 Proof of invariance

Theorem Let $\mathfrak{z}_0$ and $\mathfrak{z}_1$ be two topologically equivalent geometrical braids. Then

$$K(\mathfrak{z}_0) = K(\mathfrak{z}_1). \tag{26}$$

Proof Consider a homotopy between $\mathfrak{z}_0$ and $\mathfrak{z}_1$, i.e. consider a continuous sequence of braids $\mathfrak{z}_s$ with $0 \leq s \leq 1$ which deforms $\mathfrak{z}_0$ into $\mathfrak{z}_1$. The basic one-forms ω_{ij} will now depend on both s and t, $\omega_{ij} = \omega_{ij}(s,t)$. Let

$$\Omega_P(s) = \omega_{P_{11}P_{12}}(s,t_1) \wedge \cdots \wedge \omega_{P_{m1}P_{m2}}(s,t_m). \tag{27}$$

Consider the prism $\Delta_m \times [0,1]$. The two triangular ends of this prism correspond to the integration domains for $K(\mathfrak{z}_0)$ and $K(\mathfrak{z}_1)$. The sides consist of points where $t_1 = 0$, $t_\alpha = t_{\alpha+1}$, or $t_m = 1$. By Stoke's theorem

$$\int_{\Delta_m} \Omega_P(1) - \int_{\Delta_m} \Omega_P(0) = \int_{\Delta_m \times [0,1]} \mathrm{d}\Omega_P(s) + \int_{\text{sides}} \Omega_P(s). \tag{28}$$

First, $\mathrm{d}\Omega_P(s) = 0$ because $\mathrm{d}\omega_{ij} = 0$ for all i,j. So we only need to show that the integrals over the sides of the prism vanish.

Consider the side $t_1 = 0$. Now $\Omega_P(s)$ equals an $m-1$ form wedged with some basic form $\omega_{ij}(s,t_1)$ where

$$\omega_{ij}(s,t_1) = \mathrm{d}\lambda_{ij}(s,t_1) = \frac{\partial\lambda_{ij}}{\partial s}\mathrm{d}s + \frac{\partial\lambda_{ij}}{\partial t_1}\mathrm{d}t_1. \tag{29}$$

But on the surface $t_1 = 0$ we have $\mathrm{d}t_1 = 0$; also, since the endpoints of the strings are fixed at $t = 0$, $\partial\lambda_{ij}/\partial s = 0$. Thus $\Omega_P(s) = 0$ on this side. Similarly $\Omega_P(s) = 0$ on the side $t_m = 1$.

We are now left with the sides $t_\alpha = t_{\alpha+1}$, for example $\alpha = 1$. Let $t_1 = t_2 = \tau$. The m form $\Omega_P(s)$ is the product of the two one-forms $\omega_{P_{21}P_{22}}(s,\tau)$ and $\omega_{P_{21}P_{22}}(s,\tau)$, and some $(m-2)$ form η. For simplicity, let $(P_{11}, P_{12}) = (1,2)$. Thus

$$\Omega_P(s) = \omega_{12}(s,\tau) \wedge \omega_{P_{21}P_{22}}(s,\tau) \wedge \eta \tag{30}$$

There are three cases. First suppose that $(P_{21}, P_{22}) = (1,2)$ as well. Then

$$\Omega_P(s) = \omega_{12}(s,\tau) \wedge \omega_{12}(s,\tau) \wedge \eta = 0 \tag{31}$$

by the antisymmetry of the wedge product at equal times.

Secondly, suppose that $(P_{21}, P_{22}) = (3, 4)$, i.e. there are no strings in common. Here we use rule one for combining Gauss diagrams equation 19. The Kontsevich integral sums over all possible matrices of indices in $\mathbb{P}_{mn}$. Thus there will also be a term with indices given by P' whose first two rows are the reverse of P, i.e. $(3, 4), (1, 2)$. By rule one these give equivalent diagrams. Summing the two gives

$$\begin{aligned} \Omega_P(s)D(P) + \Omega_{P'}(s)D(P') &= (\omega_{12} \wedge \omega_{34} + \omega_{34} \wedge \omega_{12}) \wedge \eta \; D(P) && (32) \\ &= 0. && (33) \end{aligned}$$

Third, suppose that there is precisely one string in common. Here we must use rule two equation 23. For definiteness, we let $P = P_1$ with $(i, j, k) = (1, 2, 3)$, so we have

$$\Omega_{P_1}(s)D(P_1) = \omega_{12}(s, \tau) \wedge \omega_{23}(s, \tau) \wedge \eta \; D(P_1). \quad (34)$$

The Kontsevich integral will also sum over matrices $P_2, \ldots, P_6$ with permutations of $(1, 2, 3)$. Let $D_1 = D(P_1)$, $\Omega_1 = \Omega_{P_1}(s)$, etc. We obtain

$$\begin{aligned} \Omega_1 D_1 + \Omega_4 D_4 &= \omega_{12} \wedge \omega_{23} \wedge \eta \; D_1 + \omega_{23} \wedge \omega_{12} \wedge \eta \; D_4 && (35) \\ &= \omega_{12} \wedge \omega_{23} \wedge \eta \; (D_1 - D_4). && (36) \end{aligned}$$

The sums $\Omega_2 D_2 + \Omega_5 D_5$ and $\Omega_3 D_3 + \Omega_6 D_6$ are similar. Using rule two, we need only consider the combination of diagrams $D_1 - D_4$. The result is

$$\sum_{q=1}^{6} \Omega_q D_q = (\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{13} + \omega_{13} \wedge \omega_{12}) \wedge \eta \; (D_1 - D_4). \quad (37)$$

But by the Arnol'd identity equation 16 the two-form in brackets vanishes.

In all three cases, the sum of terms integrated on the side $t_1 = t_2$ vanishes. In fact, all side integrals vanish. Going back to equation 28, we see that

$$\int_{\Delta_m} \Omega_P(1) - \int_{\Delta_m} \Omega_P(0) = 0, \quad (38)$$

proving the theorem.

3 Hamiltonians generated by topological invariants

Consider two strings a and b, with complex winding number $\lambda_{ab}(t) = (\log(b - a) - \log(b_0 - a_0)/(2\pi \mathrm{i})$. The real part gives the winding number, while the imaginary part merely keeps track of the log of the distance between the two strings. As the imaginary part seems less topologically interesting, let us set it to be a constant. More precisely, we can use it as a Hamiltonian. We can also obtain Hamiltonians from the higher order invariants [7].

It will be useful at this point to write down Hamilton's equations in complex coordinates. Given n coordinates q_j, $j = 1 \ldots n$, with conjugate momenta p_j, we let $z_j = q_j + \mathrm{i}p_j$ and $\bar{z}_j = q_j - \mathrm{i}p_j$. We then convert to complex coordinates $(z_j, \bar{z}_j)$. Hamilton's equations become

$$\frac{dz_j}{dt} = -2\mathrm{i}\frac{\partial H}{\partial \bar{z}_j} \tag{39}$$

together with the complex conjugate of this equation. Next suppose that we start with an analytic function $F(z_1, \ldots, z_n)$ and let the imaginary part be the Hamiltonian:

$$F = -K + \mathrm{i}H \tag{40}$$

for some real K. Then Hamilton's equations give

$$\frac{dz_j}{dt} = \frac{\partial \bar{F}}{\partial \bar{z}_j}. \tag{41}$$

The velocity vector dz_j/dt is in essence equal to the gradient of K (to see this, one must first raise ∇K into a contravariant vector using the Euclidean metric in complex coordinates [7]). Thus the particles are pushed in the direction of maximal increase of K.

If $n = 2$ and $F = -\lambda_{ab}$, then the two particles will circle each other at constant distance. $H = -(\log|b-a|)/2\pi$ is in fact the Hamiltonian for two point vortices with unit vorticity (the minus sign ensures that the motion is anticlockwise).

Next consider $n = 3$. Let $\widetilde{\lambda}_{ab} = \lambda_{ab} + \lambda_{ab0}$ where λ_{ab0} is some initial value, for example $\log(b(0) - a(0))$ for some initial branch of the logarithm. Also let $F = \Psi_{abc}(t)$ (see equation 15), where

$$\Psi_{abc}(t) = \frac{1}{2}\int_0^t (\widetilde{\lambda}_{ab} - \widetilde{\lambda}_{bc})\omega_{ca} + (\widetilde{\lambda}_{bc} - \widetilde{\lambda}_{ca})\omega_{ab} + (\widetilde{\lambda}_{ca} - \widetilde{\lambda}_{ab})\omega_{bc}. \tag{42}$$

The Hamiltonian generated from this quantity is symmetric to translations, rotations, and scalings $z- > \lambda z$. Thus by Noether's theorem there are corresponding conserved quantities, e.g. centre of mass $a + b + c$ and mean

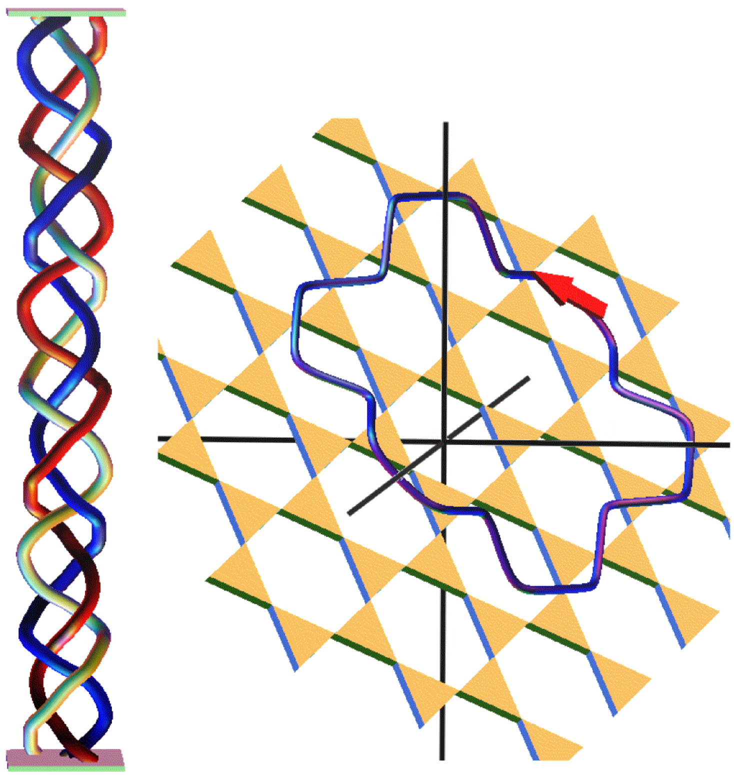


Figure 3: A braid generated by the second order invariant Ψ_{abc}. The initial conditions are $a = 0$, $b = 1$, and $c = 2$, with $2\pi\mathrm{i}(\lambda_{ab0}, \lambda_{bc0}, \lambda_{ca0}) = (\pi, 3\pi, \log 2 + \pi\mathrm{i})$.

square scale $|a|^2 + |b|^2 + |c|^2$. These are sufficient to make the dynamics integrable. Figure 1 and figure 2 show the braids generated for initial conditions $a = 0$, $b = 1$, and $c = 2$, with $\lambda_{ab0} = \lambda_{bc0} = 0$, $2\pi \mathrm{i} \lambda_{ac0} = \log 2 + \pi \mathrm{i}$. Figure 3 shows a space-time diagram and a winding diagram for initial conditions on a higher branch of the Riemann surface.

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