1 The Semi-classical approximation

Consider $V \in L^{\frac{d+2}{2}}(\mathbb{R}^d)$. Recall that $\frac{d+2}{2}$ is the exponent in the Lieb-Thirring inequality. We will study the Friedrichs extension of the operator

$$H_h = -h^2\Delta + V$$

on $C_0^2(\mathbb{R}^d)$ (or with the above assumption on V we might equally well take the domain to be the Schwartz space $\mathcal{S}(\mathbb{R}^d)$). Here $h > 0$ is a (semi-classical) parameter which we will let tend to zero.

Denote by $\mu_n(h)$ the min-max values of H_h. Our goal is to prove the following theorem.

Theorem 1 (semi-classics for eigenvalue sum)**.** *With the assumptions above we have the Weyl law for the sum of the negative eigenvalues*

$$\lim_{h\to 0} h^d \sum_{n=1}^{\infty} [\mu_n(h)]_- = (2\pi)^{-d} \iint [p^2 + V(u)]_- dpdu. \tag{1}$$

In order to prove this theorem we introduce the *coherent state* quantization.

1.1 Coherent state quantization

Let $g \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function satisfying the properties

(i) $g(x) \geq 0$, $g(x) = g(-x)$.

(ii) $\int g^2 = 1$.

For $s > 0$ let $g_s(x) = s^{-d/2} g(x/s)$. Then g_s also satisfies (i)and (ii). Note that g_s localizes on a scale s. In the next section we will choose g to have compact support in the unit ball centered at the origin, in which case g_s is supported in the ball of radius s centered at the origin. The Fourier transform of g is

$$\widehat{g}_s(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi x} g_s(x)dx = s^{d/2}\widehat{g}(xs).$$

The Fourier transform $\widehat{g}$ is also a Schwartz function. If g has compact support $\widehat{g}$ will not have compact support (it is, in fact, a real analytic function).

Since $g(x) = g(-x)$ it is straightforward to see that $\widehat{g}$ is real and satisfies $\widehat{g}(\xi) = \widehat{g}(-\xi)$. Moreover, $\int \widehat{g}^2 = 1$.

For $u, p \in \mathbb{R}^d$ we define the function

$$f_{u,p}(x) = g_s(x-u)e^{ipx} \in \mathcal{S}(\mathbb{R}^d).$$

We will use $f_{u,p}$ to localize in the Hilbert space $L^2(\mathbb{R}^d)$ near a region associated to the classical phase space point (u, p). Note that $f_{u,p}$, indeed, localizes x on a scale s from u. The Fourier transform of this function is

$$\widehat{f}_{u,p}(\xi) = (2\pi)^{-d/2} \int e^{-ix\xi} f_{u,p}(x)dx = \widehat{g}_s(\xi - p)e^{-i\xi u}e^{ipu}.$$

Note the similar form (up to the overall phase e^{ipu}) of the expressions of $f_{u,p}$ and $\widehat{f}_{u,p}$. Hence $\widehat{f}_{u,p}$ localizes in momentum space on a scale s^{-1} near p.

The functions $f_{u,p}$ are said to be *coherent states* more accurately the coherent state is the one-dimensional projection onto $f_{u,p}$, i.e.,

$$\Pi_{u,p} = |f_{u,p}\rangle\langle f_{u,p}|,$$

using Dirac notation.

In the classical definition of coherent states the function g is chosen to be the Gaussian

$$g(x) = \pi^{-d/4}e^{-x^2/2}, \tag{2}$$

but we will not restrict attention to this case.

If A is an operator (possibly unbounded) with domain containing $\mathcal{S}(\mathbb{R}^d)$ we define the *covariant* or *lower* symbol of A as the function

$$\check{A}(u, p) = \langle f_{u,p}, Af_{u,p}\rangle \tag{3}$$

As an example, consider $G : \mathbb{R}^d \to \mathbb{R}$. The lower symbol of the multiplication operator $G(x)$ (assuming it maps $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$) is then

$$\begin{aligned}\langle f_{u,p}, G(x)f_{u,p}\rangle &= \int_{\mathbb{R}^d} G(x)g_s^2(x-u)dx = \int_{\mathbb{R}^d} G(x)g_s^2(u-x)dx \\ &= G * g_s^2(u) \qquad (4)\end{aligned}$$

where we have introduced the *convolution*

$$f_1 * f_2(u) = \int f_1(u-x)f_2(x)dx = \int f_1(x)f_2(u-x)dx.$$

Likewise, going to momentum space we see that

$$\begin{aligned} \langle f_{u,p}, G(-i\nabla) f_{u,p}\rangle &= \langle \widehat{f}_{u,p}, G(\xi)\widehat{f}_{u,p}\rangle \\ &= G * \widehat{g}_s^2(p). \end{aligned} \tag{5}$$

In particular if $G_0(\xi) = 1$, $G_1(\xi) = \xi_1$ and $G_2(\xi) = \xi^2$ we find

$$G_0 * \widehat{g}_s^2(p) = 1 \tag{6}$$

$$G_1 * \widehat{g}_s^2(p) = \int (p_1 - \xi_1)\widehat{g}_s(\xi)d\xi = p_1 \tag{7}$$

$$G_2 * \widehat{g}_s^2(p) = \int (p-\xi)^2 \widehat{g}_s^2(\xi)d\xi = \int (p^2 - 2p\xi + \xi^2)\widehat{g}_s^2(\xi)d\xi \tag{8}$$

$$\begin{aligned} &= p^2 + \int \xi^2 \widehat{g}_s^2(\xi)d\xi = p^2 + \int (\nabla g_s)^2 \\ &= p^2 + s^{-2}\int (\nabla g)^2, \end{aligned} \tag{9}$$

where we used that $\int \xi \widehat{g}_s^2(\xi)d\xi = 0$ since $\widehat{g}_s(-\xi) = \widehat{g}_s(\xi)$ and in the last equality we used that $\int (\nabla g_s)^2 = s^{-2}\int (\nabla g)^2$. Thus,

$$\langle f_{u,p}, f_{u,p}\rangle = 1 \tag{10}$$

$$\langle f_{u,p}, x_1 f_{u,p}\rangle = u_1 \tag{11}$$

$$\langle f_{u,p}, -i\partial_{x_1} f_{u,p}\rangle = p_1 \tag{12}$$

$$\langle f_{u,p}, -\Delta f_{u,p}\rangle = p^2 + s^{-2}\int (\nabla g)^2. \tag{13}$$

Combining (4) and (13) we see that the lower symbol of the operator H_h is

$$\check{H}_h(u,p) = \langle f_{u,p}, H_h f_{u,p}\rangle = h^2p^2 + V * g_s^2(u) + s^{-2}\int (\nabla g)^2. \tag{14}$$

Thus the lower symbol of H_h is not quite the corresponding classical function $h^2p^2 + V(u)$.

On the other hand, given a function $a : \mathbb{R}^{2d} \to \mathbb{R}^d$ such that $a\Phi \in L^1(\mathbb{R}^{2d})$ for all $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ we may define the *coherent state quantization* to be the operator $Op(a)$ given on the domain $\mathcal{S}(\mathbb{R}^d)$ by

$$Op(a)f(x) = (2\pi)^{-d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} a(u,p) f_{u,p}(x)\langle f_{u,p}, f\rangle du dp. \tag{15}$$

If $f \in \mathcal{S}(\mathbb{R}^d)$ then $(u,p) \mapsto \langle f_{u,p}, f\rangle$ defines a function in $\mathcal{S}(\mathbb{R}^{2d})$ and by assumption the above integral hence defines a function $op(a)f \in L^2(\mathbb{R}^d)$. For the expectation values we have

$$\langle f, Op(a)f\rangle = (2\pi)^{-d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} a(u,p)|\langle f, f_{u,p}\rangle|^2 dudp, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

We say that a is the *upper* or *contravariant* symbol of $Op(a)$. In general a quantization is a map from functions on classical phase space (here $\mathbb{R}^{2d}$) to operators on Hilbert space. We shall not discuss quantizations in general. When the classical choice (2) is made the quatization introduced here is sometimes also called *anti-Wick quantization* (see Problem 1 for an explanation of this name). A commonly used quantization is Weyl quantization, which we will not discuss here, but see Problem 2 for the one-dimensional case.

We will also write

$$Op(a) = (2\pi)^{-d} \iint a(u,p)\Pi_{u,p} dudp$$

where this integral over operators is to be understood in the sense given in (15), i,e., the strong operator sense.

Consider the cases $a(u,p) = G(u)$ and $a(u,p) = G(p)$. In the first case we have by the unitarity of the Fourier transform

$$\begin{aligned} \langle f, Op(G(u))f\rangle &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(u) \left|(2\pi)^{-d/2} \int f(x) g_s(x-u) e^{-ipx} dx\right|^2 dudp \\ &= \int_{\mathbb{R}^d} |f(x)|^2 G(u) g_s^2(x-u) dudx, \end{aligned}$$

i.e., we get the multiplication operator.

$$Op(G(u)) = G * g_s^2(x). \tag{16}$$

Likewise,

$$Op(G(p)) = G * \widehat{g}_s^2(-i\nabla). \tag{17}$$

In particular, we get using (6-9)

$$\begin{aligned} Op(1) &= (2\pi)^{-d}\iint \Pi_{u,p}\,du\,dp = 1. && (18)\\ Op(u_1) &= x_1 && (19)\\ Op(p_1) &= -i\partial_{x_1} && (20)\\ Op(p^2) &= -\Delta + s^{-2}\int (\nabla g)^2 && (21)\end{aligned}$$

Combining (4) and (13) we see that

$$Op(h^2p^2 + V(u)) = -h^2\Delta + V * g_s^2(x) + s^{-2}\int (\nabla g)^2, \tag{22}$$

as an operator on $\mathcal{S}(\mathbb{R}^d)$. Again we do not quite get H_h.

For later use let us note that from

$$\begin{aligned} |\nabla(g_sf)|^2 &= |\nabla f|^2|g_s|^2 + |f|^2|\nabla g_s|^2 + (\overline{f}\nabla f + f\overline{\nabla f})g_s\nabla g_s\\ &= |\nabla f|^2|g_s|^2 + |f|^2|\nabla g_s|^2 + \frac{1}{2}\nabla|f|^2\nabla(g_s^2)\end{aligned}$$

we obtain

$$\begin{aligned} \langle f, Op(\mathbf{1}_{\{|u|<R\}}p^2)f\rangle &= \int_{|u|<R}\int |\nabla_x(g_s(x-u)f(x))|^2\,dx\,du\\ &= \int_{|u|<R}\int g_s(x-u)^2|\nabla f(x))|^2\,dx\,du\\ &\quad + \int |f(x)|^2W_{s,R}(x)\,dx\\ &\le \int |\nabla f(x))|^2dx + \int |f(x)|^2W_{s,R}(x)\,dx, && (23)\end{aligned}$$

where

$$W_{s,R}(x) = \int_{|u|<R} (\nabla g_s)^2(x-u) - \frac{1}{2}\Delta(g_s^2)(x-u)\,du \tag{24}$$

In particular, we have as an operator on $\mathcal{S}(\mathbb{R}^d)$

$$Op(\mathbf{1}_{\{|u|<R\}}h^2p^2 + V(u)) \le -h^2\Delta + V * g_s^2 + h^2W_{s,R}. \tag{25}$$

1.2 Eigenvalue sums and quantization

We consider a semi-bounded operator A defined on $\mathcal{S}(\mathbb{R}^d)$ and denote $\mu_j(A)$, $j = 1, 2, \ldots$ its min-max values.

Lemma 2. *If A is an operator as above then*

$$\sum_{j=1}^{\infty} \mu_j(A)_- = \inf \Big\{ \sum_n \nu_n \langle \phi_n, A\phi_n \rangle \ \Big| \{\phi_n\} \subset \mathcal{S}(\mathbb{R}^d), \ \textit{orthonormal (not necessarily finite) family and } 0 \le \nu_n \le 1, \ \textit{for all } n \Big\}.$$

Proof of $\le$: Given an orthonormal family $\{\phi_n\}_{n=1}$ in $\mathcal{S}(\mathbb{R}^d)$ and corresponding numbers $0 \le \nu_n \le 1$. We may assume that all $\langle \phi_n, H\phi_n \rangle < 0$ otherwise we remove the non-negative terms. We may then further assume that all $\nu_n = 1$ as will make the sum as small as possible. Even if the expectation values $\langle \phi_n, A\phi_n \rangle$ happen to be ordered increasingly it does not in general hold that $\mu_n(A)$ is less than $\langle \phi_n, A\phi_n \rangle$ for all $n = 1, 2, \ldots$. We can however conclude that

$$\sum_{j=1}^{n} \mu_j(A) \le \sum_{j=1}^{n} \langle \phi_j, A\phi_j \rangle, \tag{26}$$

for all $n = 1, \ldots$, which proves the claim. Let us show (26) by induction on n for all orthonormal families. The case $n = 1$ follows directly from the Min-max Theorem. Assume (26) holds for $n - 1$. Choose $\psi_n \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ normalized such that

$$\langle \psi_n, A\psi_n \rangle = \max\{\langle \psi, A\psi \rangle \mid \|\psi\| = 1, \ \psi \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}\}.$$

Then by the Min-max Theorem $\mu_n \le \langle \psi_n, A\psi_n \rangle$. Now supplement ψ_n to an orthonormal basis $\psi_1, \ldots, \psi_n$ for $\operatorname{span}\{\phi_1, \ldots, \phi_n\}$. By unitary invariance of the trace we then find

$$\sum_{j=1}^{n} \langle \phi_j, A\phi_j \rangle = \sum_{j=1}^{n} \langle \psi_j, A\psi_j \rangle \ge \sum_{j=1}^{n-1} \langle \psi_j, A\psi_j \rangle + \mu_n \ge \sum_{j=1}^{n-1} \mu_j + \mu_n$$

where we have used the induction assumption in the last inequality. This proves (26).

Proof of $\ge$: By the spectral theorem used on the Friedrichs extension of A we see that for a given integer N we can find an orthonormal family $\{\phi_n\}_{n=1}^N$

in $\mathcal{S}(\mathbb{R}^d)$ such that the numbers $\langle\phi_j, A\phi_j\rangle$, are arbitrarily close to $\mu_j(A)$ for $j = 1, \ldots, N$. It is therefore clear that $\sum_{j=1}^{\infty}\mu_j(A)_-$ can be approximated arbitrarily well (also when it is $-\infty$) by $\sum_{n=1}^{N}\langle\phi_n, A\phi_n\rangle$, i.e., a sum as above with $\nu_n = 1$. □

Lemma 3. *If A is an operator as above then for any $M : \mathbb{R}^{2d} \to \mathbb{R}$ with compact support and $0 \le M(u,p) \le 1$ we have in terms of the lower symbol $\check{A}(u,p) = \langle f_{u,p}, Af_{u,p}\rangle$*

$$\sum_{j=1}^{\infty}\mu_j(A)_- \le (2\pi)^{-d}\iint M(u,p)\check{A}(u,p)dudp.$$

Proof. We will use Lemma 2. Instead of directly constructing ν_n and ϕ_n we will construct the operator $Kf = \sum_n \nu_n\langle\phi_n, f\rangle\phi_n$.

In fact, we use the quantization method and define $K = Op(M)$. Then K satisfies

(i) $0 \le K \le 1$. In particular, the max-min values of K satisfy $0 \le \nu_j(K) \le 1$ for all $j = 1, 2, \ldots$.

(ii) K is trace class, i.e.,
$$\sum_{j=1}^{\infty}\nu_j(K) < \infty.$$

To see (i) note that

$$\begin{aligned}\langle f, Kf\rangle &= (2\pi)^{-d}\iint M(u,p)|\langle f_{u,p}, f\rangle|^2 dudp\\ &\le (2\pi)^{-d}\iint |\langle f_{u,p}, f\rangle|^2 dudp = \|f\|^2\end{aligned}$$

by (18).

To see (ii) let $\{\psi_j\}$ be any orthonormal family in $L^2(\mathbb{R}^d)$ then

$$\begin{aligned}\sum_j\langle\psi_j, K\psi_j\rangle &= (2\pi)^{-d}\sum_j\iint M(u,p)|\langle f_{u,p}, \psi_j\rangle|^2 dudp\\ &\le (2\pi)^{-d}\iint M(u,p)dudp < \infty.\end{aligned}$$

It follows that K cannot have positive essential spectrum and hence that all positive max-min values are eigenvalues of finite multiplicity. In addition zero might or might not be an eigenvalue and it might or might not be a max-min value (the latter will happen if there are only finitely many positive eigenvalues).

Now choosing $\{\psi_j\}$ to be an orthonormal basis of $L^2(\mathbb{R}^d)$ consisting of eigenfunctions for K we see that K is trace class with

$$\sum_{j=1}^{\infty} \nu_j(K) = (2\pi)^{-d} \iint M(u,p)dudp.$$

It is not actually important in the following exactly what the trace of K is.

Let us continue to denote by $\{\psi_j\}$ an orthonormal basis of eigenfunctions of K. It is not always possible to order these such that the corresponding eigenvalues are the max-min values $\nu_j(K)$ (why not?). Let us denote the eigenvector corresponding to $\nu_j(K)$ by ϕ_j. The $\{\phi_j\}$ may be chosen to form a subset of $\{\psi_j\}$, but there could be additional zero-eigenvalue eigenfunctions.

Since

$$\nu_j(K)\phi_j(x) = K\phi_j(x) = (2\pi)^{-d} \iint M(u,p)f_{u,p}(x)\langle f_{u,p}, \phi_j\rangle dudp,$$

it follows that $\nu_j(K)\phi_j \in \mathcal{S}(\mathbb{R}^d)$ for all j. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} \nu_j(K)\langle \phi_j, A\phi_j\rangle &= \sum_{j=1}^{\infty} \langle K\psi_j, A\psi_j\rangle \\
&= \sum_{j=1}^{\infty} (2\pi)^{-d} \iint M(u,p)\langle \psi_j, f_{u,p}\rangle\langle f_{u,p}, A\psi_j\rangle dudp \\
&= \sum_{j=1}^{\infty} (2\pi)^{-d} \iint M(u,p)\langle \psi_j, f_{u,p}\rangle\langle Af_{u,p}, \psi_j\rangle dudp.
\end{aligned}$$

Since A maps $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ and A is a closable operator (as it is symmetric) it follows from the Closed graph Theorem that A is in fact continuous from $\mathcal{S}(\mathbb{R}^d)$ (with its Fréchet topology) to $L^2(\mathbb{R}^d)$. Indeed, if $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ then $f_n \to f$ in L^2 and if moreover $Af_n \to \psi$ in L^2 it follows from the closability of A that $Af = \psi$. As $(u,p) \mapsto f_{u,p}$ is a continuous map from $\mathbb{R}^{2d}$ to $\mathcal{S}(\mathbb{R}^d)$ we conlude that $(u,p) \mapsto \|Af_{u,p}\|$ is continuous. It follows that we can interchange the sum and integral above (as M has compact support). Thus

$$\begin{aligned}\sum_{j=1}^{\infty}\nu_j(K)\langle\phi_j, A\phi_j\rangle &= (2\pi)^{-d}\iint M(u,p)\langle f_{u,p}, Af_{u,p}\rangle dudp\\ &= (2\pi)^{-d}\iint M(u,p)\check{A}(u,p)dudp.\end{aligned}$$

The lemma follows from Lemma 2. □

Theorem 4. *Let A be a semi-bounded operator defined on $\mathcal{S}(\mathbb{R}^d)$. Then*

$$\sum_{j=1}^{\infty}\mu_j(A)_- \le (2\pi)^{-d}\iint \check{A}(u,p)_- dudp. \tag{27}$$

Conversely, if $a : \mathbb{R}^{2d} \to \mathbb{R}^d$ satisfies that $a\Phi \in L^1(\mathbb{R}^d)$ for all $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ then

$$\sum_{j=1}^{\infty}\mu_j(Op(a)) \ge (2\pi)^{-d}\iint a(u,p)_- dudp. \tag{28}$$

Included in the statement is that the operator $Op(a)$ is bounded below (and thus the min-max values make sense) if the integral on the left is finite.

Proof. The estimate in (27) follows immediately from Lemma 3 by choosing a monotone increasing sequence of functions M_n of compact support approaching the characteristic function of the set where $\check{A}$ is negative.

Since $op(a)$ is defined on $\mathcal{S}(\mathbb{R}^d)$ it follows from Lemma 2 that in order to establish (28) we must show that for any orthonormal family $\phi_1, \dots, \phi_n$ in $\mathcal{S}(\mathbb{R}^d)$. we have

$$\sum_{j=1}^{n}\langle\phi_j, op(a)\phi_j\rangle \ge (2\pi)^{-d}\iint a(u,p)_-.$$

This follows since

$$\begin{aligned}\sum_{j=1}^{n}\langle\phi_j, op(a)\phi_j\rangle &= (2\pi)^{-d}\iint a(u,p)\sum_{j=1}^{n}|\langle\phi_j, f_{u,p}\rangle|^2 dudp\\ &\ge (2\pi)^{-d}\iint a(u,p)_-\sum_{j=1}^{n}|\langle\phi_j, f_{u,p}\rangle|^2 dudp\\ &\ge (2\pi)^{-d}\iint a(u,p)_- dudp\end{aligned}$$

since

$$0 \leq \sum_{j=1}^{n} |\langle \phi_j, f_{u,p}\rangle|^2 \leq 1.$$

for all (u,p). $\square$

1.3 Proof of semiclassical theorem

We are now ready to prove Theorem 1. In this section we will choose g to have compact support in the unit ball centered at the origin, such that g_s has support in the ball of radius s.

Proof of upper bound in (1). We use Lemma 3 and choose $M(u,p)$ to be the characteristic function of the set

$$\{(u,p) \mid |u| \leq R,\ h^2p^2 + V(u) \leq 1\}$$

for some $R > 0$ to be chosen below. From (14) we then find that

$$\begin{aligned}
\sum_{n=1}^{\infty} [\mu_n(h)]_- &\leq (2\pi)^{-d} \iint\limits_{h^2p^2+V(u)\leq 0,\ |u|\leq R} h^2p^2 + V * g_s^2(u) + Ch^2 s^{-2} du dp \\
&= (2\pi)^{-d} \iint (h^2p^2 + V(u))_- du dp + E_1 + E_2 + E_3
\end{aligned}$$

where the error terms are

$$\begin{aligned}
E_1 &= -(2\pi)^{-d} \iint\limits_{|u|\geq R} (h^2p^2 + V(u))_- du dp \leq Ch^{-d} \int\limits_{|u|\geq R} |V_-(u)|^{\frac{d+2}{2}} du \\
E_2 &\leq Ch^{-d}h^2 s^{-2} \int\limits_{|u|\leq R} |V_-(u)|^{d/2} du \\
E_3 &\leq Ch^{-d} \int |V_-(u)|^{d/2} |V(u) - V * g_s^2(u)| du \\
&\leq Ch^{-d} \left(\int |V_-(u)|^{\frac{d+2}{2}} du\right)^{\frac{d}{d+2}} \left(\int |V(u) - V * g_s^2(u)|^{\frac{d+2}{2}} du\right)^{\frac{2}{d+2}}.
\end{aligned}$$

We must argue that $h^d(E_1 + E_2 + E_3)$ can be chosen arbitrarily small. To do this we will use that $V \in L^{\frac{d+2}{2}}(\mathbb{R}^d)$ and hence $V * g_s^2 \to V$ in $L^{\frac{d+2}{2}}$ as $s \to 0$.

Hence we can first choose s so small and R so large that $h^d E_3$ and $h^d E_1$ are as small as we like. Finally, since

$$\begin{aligned}\int\limits_{|u|\leq R} |V_-(u)|^{d/2}du &\leq \left(\int |V_-(u)|^{\frac{d+2}{2}}du\right)^{\frac{d}{d+2}} \left(\int\limits_{|u|\leq R} 1du\right)^{\frac{2}{d+2}} \\ &\leq CR^{\frac{2d}{d+2}}\left(\int |V_-(u)|^{\frac{d+2}{2}}du\right)^{\frac{d}{d+2}}\end{aligned}$$

we see for these choices of s and R that if h is suffiently small then $h^d E_2$ will also be as small as we like (because of the extra h^2). □

Proof of lower bound in (1). By Lemma 2 we have to show that for any orthonormal family $\phi_1, \ldots, \phi_n$ we have

$$\sum_{j=1}^{n}\langle\phi_j, H_h\phi_j\rangle \geq (2\pi)^{-d}\iint(h^2p^2 + V(u))_- dpdu - E(h),$$

where $h^d E(h) \to 0$ as $h \to 0$.

For $R > 0$ we introduce $V_R(x) = V(x)$ if $|x| < R$ and $V_R(x) = 0$ if $|x| > R$. For $0 < \delta < 1$ we then write

$$H_h = H^{(0)} + H^{(1)},$$

where

$$\begin{aligned}H^{(0)} &= -(1-\delta)h^2\Delta + V_R * g_s^2(x) + (1-\delta)h^2 W_{s,R}(x), \\ H^{(1)} &= -\delta h^2\Delta + V(x) - V_R * g_s^2(x) - (1-\delta)h^2 W_{s,R}(x),\end{aligned}$$

with $W_{s,R}$ given in (24). Note that since g_s has support in the ball of radis s we have

$$|W_{s,R}(x)| \leq Cs^{-2}\mathbf{1}_{\{|x|<R+s\}}(x). \tag{29}$$

and C depends only on g.

We will estimate $H^{(0)}$ and $H^{(1)}$ separately. We will use the Lieb-Thirring inequality to estimate $H^{(1)}$. Writing the operator as

$$H^{(1)} = (\delta h^2)\left(-\Delta + (\delta h^2)^{-1}\left(V(x) - V_R * g_s^2(x) - (1-\delta)h^2 W_{s,R}(x)\right)\right)$$

we see from the Lieb-Thirring inequality that

$$\begin{aligned}
&\sum_{j=1}^{n}\langle \phi_j, H^{(1)}\phi_j\rangle \\
&\geq -C(\delta h^2)(\delta h^2)^{-\frac{d+2}{2}}\int \left(|V(x)-V_R * g_s^2(x)| + (1-\delta)h^2 W_{s,R}(x)\right)^{\frac{d+2}{2}}\,dx \\
&= -C\delta^{-d/2}h^{-d}\||V(x)-V_R * g_s^2(x)| + (1-\delta)h^2 W_{s,R}(x)\|_{\frac{d+2}{2}}^{\frac{d+2}{2}} \\
&\geq -C\delta^{-d/2}h^{-d}\left(\|V-V_R * g_s^2\|_{\frac{d+2}{2}} + (1-\delta)h^2\|W_{s,R}\|_{\frac{d+2}{2}}\right)^{\frac{d+2}{2}} \\
&\geq -Ch^{-d}\left(\delta^{-d/2}\|V-V_R * g_s^2\|_{\frac{d+2}{2}} + C\delta^{-d/2}h^2 s^{-2}(R+s)^{\frac{2d}{d+2}}\right)^{\frac{d+2}{2}}, \qquad (30)
\end{aligned}$$

where we have used the triangle inequality for the norm $L^{\frac{d+2}{2}}$ and the estimate in (29).

To estimate $H^{(0)}$ we note from (25) that

$$H^{(0)} = Op\left(((1-\delta)h^2p^2 + V(u))\mathbf{1}_{\{|u|<R\}}\right).$$

Hence by Lemma 2 and Theorem 4 we get

$$\begin{aligned}
\sum_{j=1}^{n}\langle \phi_j, H^{(0)}\phi_j\rangle &\geq (2\pi)^{-d}\iint_{|u|<R}((1-\delta)h^2p^2 + V(u))_- du dp \\
&\geq (2\pi)^{-d}\iint((1-\delta)h^2p^2 + V(u))_- du dp \\
&\geq h^{-d}(1-\delta)^{-d/2}(2\pi)^{-d}\iint(p^2 + V(u))_- du dp \\
&\geq h^{-d}(2\pi)^{-d}\iint(p^2 + V(u))_- du dp \\
&\quad -C((1-\delta)^{-d/2}-1)h^{-d}\int |V_-(u)|^{\frac{d+2}{2}}du.
\end{aligned}$$

To finish the argument we first choose δ so close to 0 that $(1-\delta)^{-d/2}$ is as close to 1 as we please. We then choose R and s such that the first term in the parenthesis in (30) is as small as we like and finally we can ensure that the last term in the parenthesis in (30) is as small as we like when h is small enough. $\square$

1.4 Problems

Problem 1 Consider the case $d = 1$ and introduce the annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), \quad a^* = \frac{1}{\sqrt{2}}(x - \frac{d}{dx}).$$

(a) Show that these operators satisfy the canonical commutation relation when acting on $\mathcal{S}(\mathbb{R})$

$$aa^* - a^*a = 1.$$

(b) Show that if g is chosen in accordance with (2) then

$$af_{u,p} = \frac{1}{\sqrt{2}}(u + ip)f_{u,p},$$

i.e., $f_{u,p}$ is an eigenvector of a with eigenvalue $\frac{1}{\sqrt{2}}(u + ip)$.

(c) Use (b) to show that

$$a\Pi_{u,p} = \frac{1}{\sqrt{2}}(u + ip)\Pi_{u,p}, \quad \Pi_{u,p}a^* = \frac{1}{\sqrt{2}}(u - ip)\Pi_{u,p}$$

(d) Use (c) to show that

$$Op((u + ip)^n(u - ip)^m) = 2^{(n+m)/2}a^n a^{*m}.$$

A product of creation and annihilation operators is said to be normal or Wick ordered if all creation operators are to the left of all annihilation operators. Conversely, if as above all creation operators are to the right of all annihilation operators we say that the product is anti-normal or anti-Wick ordered.

(e) Show that for a normal ordered product $a^{*m}a^n$ the lower symbol is

$$\langle f_{u,p}, a^{*m}a^n f_{u,p}\rangle = 2^{-(n+m)/2}(u + ip)^n(u - ip)^m$$

Problem 2 We use the same assumptions as in Problem 1. The Weyl quantization is given by

$$Op^{\mathrm{W}}(a)f(x) = (2\pi)^{-d} \iint a\left(\frac{x+y}{2}, p\right) e^{ip(x-y)} f(y)dydp.$$

- Find $Op^{\mathrm{W}}(u^2 + p^2)$.
- Find $Op^{\mathrm{W}}((u+ip)^n(u-ip)^m)$ expressed with creation and annihilation operators.