MATH6502 Example Sheet 4. Hand in all questions from section A.
Cover sheet with DEPARTMENT/TUTOR/YOUR NAME & signed.
Due into Maths room 6.10 by 2pm on Wednesday 12 November.

Section A

Just one question this week, but it is longer than usual. On the other hand, you do have reading week to work on it. Note that you can attempt all the parts; (b) and (c) don't depend on being able to do (a) and (c) doesn't depend on (b).

1. Consider Laplace's equation in polar coordinates:

$$r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0.$$

   (a) Show that the general solution to this equation in a ring-shaped region $r_a \leq r \leq r_b$ is given by

$$f(r, \theta) = C + D \ln r + \sum_n (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n}).$$

   (b) Using the solution above, find the specific solution to Laplace's equation which satisfies

$$f(r_a, \theta) = 0 \qquad f(r_b, \theta) = 1.$$

   (c) Now consider an infinite region $r > r_a$. The general solution (eliminating terms which are unbounded as $r \to \infty$) is

$$f(r, \theta) = C + \sum_n (a_n \cos n\theta + b_n \sin n\theta) r^{-n}.$$

   Use this to find the steady temperature outside a disc which has a constant temperature gradient inside it, $T = x$, so that on the boundary we have

$$f(r_a, \theta) = r_a \cos \theta,$$

   where the far-field temperature is zero, $f(r, \theta) \to 0$ as $r \to \infty$.

Section B

More questions on separation of variables.

1. Find the steady temperature inside a square plate one of whose sides is held at temperature 1, the other three being kept at 0.
2. Find the temperature for all time (i.e. using the unsteady heat equation) for a bar of length L (with ends at $x = 0$ and L) which is initially held at a temperature $L^2 - (2x - L)^2$ and whose ends are held at temperature 0.