Handout 9 More matrix properties; the transpose

Square matrix properties

These properties only apply to a square matrix, i.e. $n \times n$.

- The *leading diagonal* is the diagonal line consisting of the entries a_{11}, a_{22}, a_{33}, $\ldots a_{nn}$.
- A *diagonal* matrix has zeros everywhere except the leading diagonal.
- The *identity* matrix $\underline{\underline{I}}$ has zeros off the leading diagonal, and 1 for each entry on the diagonal. It is a special case of a diagonal matrix, and $\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}$ for any $n \times n$ matrix $\underline{\underline{A}}$.
- An *upper triangular matrix* has all its non-zero entries on or above the leading diagonal.
- A *lower triangular matrix* has all its non-zero entries on or below the leading diagonal.
- A *symmetric matrix* has the same entries below and above the diagonal: $a_{ij} = a_{ji}$ for any values of i and j between 1 and n.
- An *antisymmetric* or *skew-symmetric matrix* has the opposite entries below and above the diagonal: $a_{ij} = -a_{ji}$ for any values of i and j between 1 and n. This automatically means the digaonal entries must all be zero.

Transpose

To transpose a matrix, we reflect it across the line given by the leading diagonal a_{11}, a_{22} etc. In general the result is a different shape to the original matrix:

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad \underline{\underline{A}}^\top = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \qquad [\underline{\underline{A}}^\top]_{ij} = \underline{\underline{A}}_{ji}.$$

- If $\underline{\underline{A}}$ is $m \times n$ then $\underline{\underline{A}}^\top$ is $n \times m$.
- The transpose of a **symmetric matrix** is itself: $\underline{\underline{A}}^\top = \underline{\underline{A}}$ (recalling that only square matrices can be symmetric).
- For an **antisymmetric matrix**, $\underline{\underline{A}}^\top = -\underline{\underline{A}}$.
- The **transpose of a product** is $(\underline{\underline{A}}\,\underline{\underline{B}})^\top = \underline{\underline{B}}^\top \underline{\underline{A}}^\top$.
- If you **add a matrix and its transpose** the result is symmetric. You can only do the addition if the matrix and its transpose are the same shape; so we need a square matrix for this.
- If you **subtract the transpose from the matrix** the result is antisymmetric.
- The transpose of a sum is the sum of the transposes (as you would expect): $\underline{\underline{A}}^\top + \underline{\underline{B}}^\top = (\underline{\underline{A}} + \underline{\underline{B}})^\top$.
- If we transpose twice we get back to where we started: $(\underline{\underline{A}}^\top)^\top = \underline{\underline{A}}$.

You can split a matrix into the sum of a symmetric and an antisymmetric matrix using the transpose:

$$\underline{\underline{A}} = \tfrac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^\top) + \tfrac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^\top).$$

The first of these is symmetric and the second, antisymmetric.