Handout 8 Matrix properties, multiplication and addition

Definition of a matrix

We define a matrix as a rectangular array of numbers: an m by n matrix $\underline{\underline{A}}$ is

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

Matrix-vector multiplication

We can multiply an m by n matrix with a vector of length n:

$$\underline{\underline{A}}\,\underline{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}.$$

The rows of the matrix must be the same length as the column of the vector.

Matrix-matrix multiplication

For an m by n matrix $\underline{\underline{A}}$ and an n by p matrix $\underline{\underline{B}}$, we can form the product $\underline{\underline{A}}\,\underline{\underline{B}}$:

$$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

The number of **columns** of $\underline{\underline{A}}$ must match the number of **rows** of $\underline{\underline{B}}$, and the product $\underline{\underline{C}}$ is an m by p matrix. The elements of $\underline{\underline{C}}$ are formed using this formula:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

The order of multiplication matters so in general $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$.
However, you will be relieved to know, $(\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}})$.

Matrix addition

Two matrices can be added if and only if they are *the same size*; then we just add the elements:

$$[\underline{\underline{A}} + \underline{\underline{B}}]_{ij} = a_{ij} + b_{ij}.$$

Two matrices are equal if and only if they are the same size and *all* their elements match.