Handout 7 Revision of ODEs (and a little new material)

Classification of ODEs

We can classify our ordinary differential equation by two main properties:

Order The order of a differential equation is the largest number of derivatives (of the function) taken.

Linearity A differential equation is linear if every term in the equation contains none or exactly one of either the function or its derivatives. There are no products of the function with itself or its derivatives. Each term has at most one power of the equivalent of f or $\mathrm{d}f/\mathrm{d}x$ or $\mathrm{d}^2 f/\mathrm{d}x^2$ or ...

and if it is linear, then three more:

Homogeneous A linear differential equation is homogeneous if every term in the equation contains exactly one of either the function or its derivatives. There are no terms without an f, $\mathrm{d}f/\mathrm{d}x$ etc.

Constant coefficients A linear differential equation has constant coefficients if the dependent variable and all the derivatives are only multiplied by constants.

Coefficients with powers matching the derivative This is a class of linear differential equations where the coefficient of the nth derivative looks like a constant times x^n.

Homogeneous linear equations.

For these equations, we can add up solutions: so if $f(x)$ is a solution and $g(x)$ is a solution, then so is $af(x) + bg(x)$ for any constants a and b.

An nth order homogeneous linear equation should have exactly n independent solutions $f_1, \ldots, f_n$ and the general solution to the equation is

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x).$$

Constant coefficients

We try an exponential function like $e^{\lambda x}$; this gives a polynomial for λ:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 5\frac{\mathrm{d}f}{\mathrm{d}x} + 6f = 0 \qquad \lambda^2 + 5\lambda + 6 = 0.$$

In this case we get two roots: $\lambda_1 = -2$ and $\lambda_2 = -3$ and the *general solution* is

$$f(x) = c_1 e^{-2x} + c_2 e^{-3x}.$$

This works fine unless there is a repeated root in the polynomial; then we use $xe^{\lambda x}$ as our extra solution.

If the polynomial has complex roots, $\lambda = a \pm ib$, then the imaginary parts give sine and cosine functions:

$$f(x) = e^{ax}[c_1 \cos bx + c_2 \sin bx].$$

Homogeneous linear equations with coefficients which are powers matching the derivative

We try a solution of the form x^m, which gives a polynomial for m:

$$x^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}f}{\mathrm{d}x} - 6f = 0 \qquad m^2 + m - 6 = 0$$

In this case we get two roots: $m_1 = 2$ and $m_2 = -3$ and the *general solution* is

$$y = c_1 x^2 + c_2 x^{-3}.$$

This works with an nth order ODE as long as the nth order polynomial for m has n different real roots. [Note, if there is a repeated root then the extra solution we need is $x^m \ln x$.]