Handout 6 Partial Differential Equations: separation of variables

This is a powerful technique for solving linear PDEs that have no mixed derivatives, i.e. nothing of the form $\partial^2 f/\partial x \partial y$.

There are 6 essential steps:

1. Assume the solution is going to be of the form $X(x)T(t)$ or $X(x)Y(y)$, etc. This is called **separable form**.
2. Substitute that form back into the PDE.
3. Divide by $X(x)T(t)$ or $X(x)Y(y)$.
4. Now each term of the equation depends on a different variable so they must both be constants.
5. For each possible value of the constant (positive, negative, zero), solve the two resulting ODEs and multiply the solutions together to give one specific solution to the PDE
6. Form the general solution of the PDE by adding linear combinations of all the specific solutions.

Example: Heat equation in one dimension

This equation governs the temperature $f(x,t)$ in a thin uniform body of conductivity κ (thin enough that temperature only varies along its length (not across the width):

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^2 f}{\partial x^2}.$$

Our process goes:

$$f(x,t) = X(x)T(t) \qquad X(x)T'(t) = \kappa X''(x)T(t) \qquad \frac{T'(t)}{T(t)} = \frac{\kappa X''(x)}{X(x)} = \kappa A.$$

Zero constant, $A = 0$

$T'(t) = 0$ so $T(t) = A_1$ and $X''(x) = 0$ so $X(x) = B_1 x + C_1$, and multiplying these and renaming the constants gives

$$f(x,t) = \alpha x + \beta.$$

Negative constant, $A = -\kappa\lambda^2$

$T'(t) = -\kappa\lambda^2 T(t)$ so $T(t) = A_2 \exp[-\kappa\lambda^2 t]$.

$X''(x) = -\kappa\lambda^2 X(x)$ so $X(x) = B_2\cos(\lambda x) + C_2 \sin(\lambda x)$. Multiplying and renaming again, we get:

$$f(x,t) = \exp[-\kappa\lambda^2 t]\left(a\cos(\lambda x) + b\sin(\lambda x)\right).$$

Positive constant, $A = \kappa\mu^2$ $T'(t) = \kappa\mu^2 T(t)$ so $T(t) = A_3\exp[\kappa\mu^2 t]$.

$X''(x) = \mu^2 X(x)$ so $X(x) = B_3\exp[\mu x] + C_3\exp[-\mu x]$. Putting them together, we have:

$$f(x,t) = \exp[\kappa\mu^2 t]\left(A\exp[\mu x] + B\exp[-\mu x]\right).$$

Note that the temperature here grows exponentially in time: these solutions are not physical!

General solution

$$f(x,t) = \alpha x + \beta + \sum_n \exp[-\kappa\lambda_n^2 t]\left(a_n\cos(\lambda_n x) + b_n\sin(\lambda_n x)\right)$$
$$+ \sum_n \exp[\kappa\mu_n^2 t]\left(A_n\exp[\mu_n x] + B_n\exp[-\mu_n x]\right).$$

Real example: heat equation in a finite length bar with cold ends

Now suppose we have a bar of length L which is initially at temperature 1 all over, and which we cool from both ends by holding the ends at temperature 0:

$$f(x,0)=1 \qquad f(0,t)=0 \qquad f(L,t)=0.$$

Left hand end Substituting $x=0$ into our general solution, we get:

$$f(0,t)=0=\beta+\sum_n a_n \exp\left[-\kappa\lambda_n^2 t\right]+\sum_n (A_n+B_n)\exp\left[\kappa\mu_n^2 t\right]$$

and forcing this for every possible t gives $b=0$, $a_n=0$ and $A_n+B_n=0$. Putting these back in makes the full solution become:

$$f(x,t)=\alpha x+\sum_n b_n \exp\left[-\kappa\lambda_n^2 t\right]\sin(\lambda_n x)+\sum_n A_n \exp\left[\kappa\mu_n^2 t\right]\left(\exp\left[\mu_n x\right]-\exp\left[-\mu_n x\right]\right).$$

Right hand end Next we look at the condition $f=0$ at $x=L$. This gives:

$$0=\alpha L+\sum_n b_n \exp\left[-\kappa\lambda_n^2 t\right]\sin(\lambda_n L)+\sum_n A_n \exp\left[\kappa\mu_n^2 t\right](\exp\left[\mu_n L\right]-\exp\left[-\mu_n L\right]).$$

Again, this has to be true for all values of t and the t-dependence of each term is different: so we end up with $\alpha=0$, $A_n=0$ and $\lambda_n L=n\pi/L$.

Now almost all the terms have disappeared and the full solution becomes:

$$f(x,t)=\sum_n b_n \exp\left[-\kappa n^2\pi^2 t/L^2\right]\sin(n\pi x/L).$$

Initial condition Now we put in $t=0$ and get:

$$1=\sum_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

which is a Fourier sine series with period $2L$ that we've seen before:

$$b_n=\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x=\frac{2}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L=\frac{2}{n\pi}\left(1-\cos(n\pi)\right)=\begin{cases}0 & n \text{ even}\\ \frac{4}{n\pi} & n \text{ odd}\end{cases}$$

We can put this back into our general solution:

$$f(x,t)=\sum_{n \text{ odd}}\frac{4}{n\pi}\exp\left[-\frac{\kappa n^2\pi^2 t}{L^2}\right]\sin\left(\frac{n\pi x}{L}\right).$$

Let's plot this function for a few values of t: