Handout 2 Maclaurin series and Taylor series

Maclaurin series

The Maclaurin series is an approximation to a function $f(x)$ near $x = 0$:

$$F(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

where we're using the notation $f^{(n)}(x)$ to mean the n-th derivative of $f(x)$.

Example: $f(x) = 1/(1-x)$

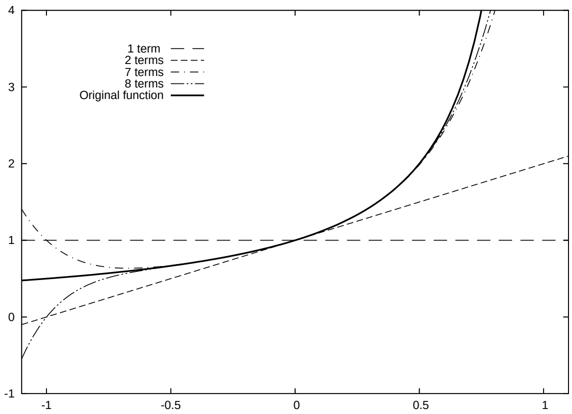
$$f(x) = \frac{1}{(1-x)} \qquad f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3} \qquad f'''(x) = \frac{6}{(1-x)^4}$$

$$f(0) = 1 \qquad f'(0) = 1$$

$$f''(0) = 2 \qquad f'''(0) = 6$$

$$F(x) = 1 + x + x^2 + x^3 + \cdots$$



This series converges (so $F(x) = f(x)$) as long as $-1 < x < 1$. It has **radius of convergence** 1.

Example: $f(x) = \sin x$

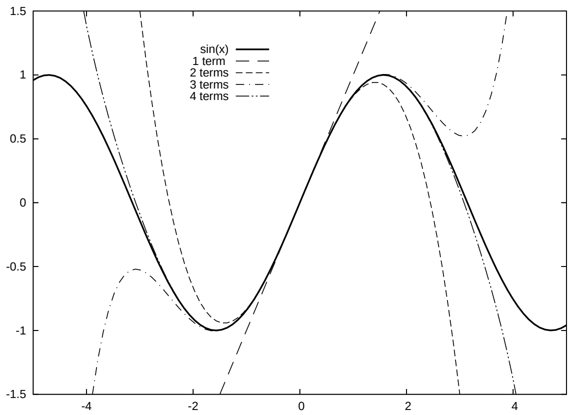
$$f(x) = \sin x \qquad f'(x) = \cos x$$

$$f''(x) = -\sin x \qquad f'''(x) = -\cos x$$

$$f(0) = 0 \qquad f'(0) = 1$$

$$f''(0) = 0 \qquad f'''(0) = -1$$

$$F(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$



In fact, this series converges for all values of x: it has an **infinite radius of convergence**.

Taylor Series

This gives an approximation for $f(x)$ near $x = a$:

$$F(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

Example: $\ln x$ **near** $x = 2$

$$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$$

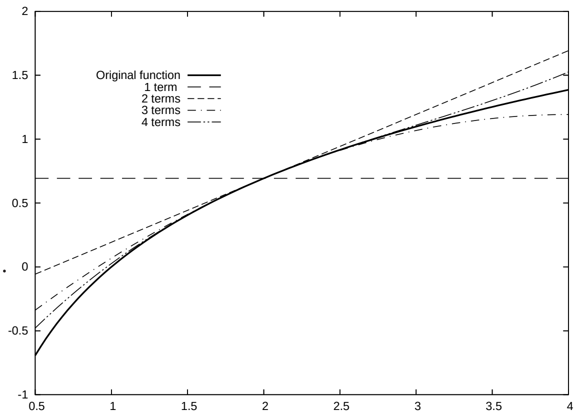
$$f''(x) = \frac{-1}{x^2} \qquad f'''(x) = \frac{2}{x^3}$$

$$f(2) = \ln 2 \quad f'(2) = 1/2$$

$$f''(2) = -1/4 \quad f'''(2) = 1/4$$

$$F(x) = \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \cdots$$

This Taylor series has radius of convergence 2: so it's OK as long as $0 < x < 4$.



Example: polynomial

Here we'll look at the intermediate terms of the Taylor series for $f(x) = x^3 - 3x$ near $x = -1$, $x = 0$ and $x = 1$.

$$f(x) = x^3 - 3x \qquad f'(x) = 3x^2 - 3 \qquad f''(x) = 6x \qquad f'''(x) = 6$$

Near $x = -1$:

$$f(-1) = 2 \qquad f'(-1) = 0 \qquad f''(-1) = -6 \qquad f'''(-1) = 6 \qquad F(x) = 2 - 3(x+1)^2 + (x+1)^3.$$

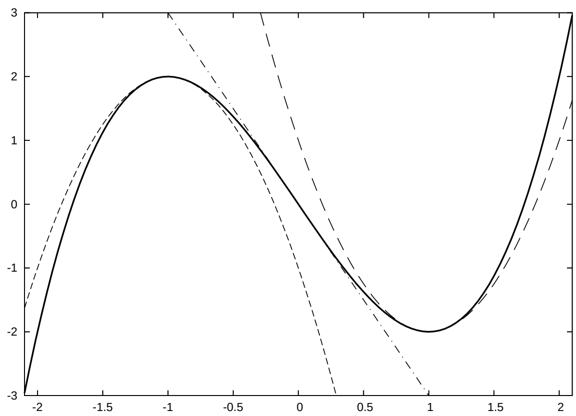
Near $x = 0$:

$$f(0) = 0 \qquad f'(0) = -3 \qquad f''(0) = 0 \qquad f'''(0) = 6 \qquad F(x) = -3x + x^3.$$

Near $x = 1$:

$$f(1) = -2 \qquad f'(1) = 0 \qquad f''(1) = 6 \qquad f'''(1) = 6 \qquad F(x) = -2 + 3(x-1)^2 + (x-1)^3.$$

All three versions of $F(x)$ are the same function; but the first or first two terms of each give three different approximations:



The straight line in the middle is $-3x$; the right hand curve is $-2 + 3(x-1)^2$; and the left hand curve is $2 - 3(x+1)^2$. The thick curve is $f(x) = F(x)$. Again, the radius of convergence here is infinite.