Handout 13 Back substitution from (generalised) row echelon form

Nonzero determinant

If the determinant of the matrix in the original system is nonzero, then

- the system has a unique solution; and
- row echelon form is an upper triangular matrix with no zeros on the diagonal and no zero rows

We simply solve from the bottom up:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -8 \end{pmatrix} \qquad \begin{array}{lcl} r_3 & : & 2z = -8 \implies z = -4 \\ r_2 & : & y - z = 6 \implies y + 4 = 6 \implies y = 2 \\ r_1 & : & x - y = 1 \implies x - 2 = 1 \implies x = 3. \end{array}$$

Zero determinant

If the determinant of the original matrix is zero then there will be at least one zero row at the bottom of the matrix in echelon form. There are two cases:

Non-zero entry in the vector on the right

If there is a row of the augmented matrix which has all zeros on the left of the bar, and a non-zero entry on the right, then there are **no solutions** to the system.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 9 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 3 & 1 & 2 & 11 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 9 \end{array}\right)$$

Zero row in the augmented matrix

If all the zero rows of the original matrix are in zero rows of the augmented matrix, then we have **an infinite number of solutions**.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 3 & 1 & 2 & 11 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The procedure to find them is:

- In the echelon-form matrix, circle the first non-zero entry in every row
- Find the columns which have no circles in (there will be the same number of these as there are zero rows). Each column corresponds to a variable.
- Assign a new name to each of these chosen variables. Then use back substitution.

In our example, we circle the 3 and the -1, which leaves column 2 without a circle in it. Now in the original matrix-vector system, column 2 corresponds to the variable x, so we set $x = \alpha$. Then we solve from the bottom up as usual:

$$-y = -3 \implies y = 3$$

$$3w + x + 2y = 11 \implies 3w + \alpha + 6 = 11 \implies w = (5 - \alpha)/3.$$

So we have found the solution $w = (5 - \alpha)/3$, $x = \alpha$ and $y = 3$ for any α: we have found the infinite number of solutions. Here are a few (found by picking a value of α):

$$\begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} 5/3 \\ 0 \\ 3 \end{pmatrix} \quad \begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} w \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$