Handout 10 Determinants

Consider an $n \times n$ matrix $\underline{\underline{A}}$. A very important property of any such square matrix is its **determinant**. We write it as

$$\det(\underline{\underline{A}}) \quad \text{or} \quad |\underline{\underline{A}}| \quad \text{or} \quad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- For a 2×2 matrix:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$
- The **minor** of element a_{ij} of $\underline{\underline{A}}$ is the determinant of $\underline{\underline{M}}_{ij}$, where $\underline{\underline{M}}_{ij}$ is the $(n-1) \times (n-1)$ matrix defined by deleting row i and column j from $\underline{\underline{A}}$.
- The **cofactor** C_{ij} of element a_{ij} of $\underline{\underline{A}}$ is $(-1)^{i+j}$ times the minor: $C_{ij} = (-1)^{i+j} \det(\underline{\underline{M}}_{ij})$.
- Expand on row i or column j to calculate the determinant:
$$\det(\underline{\underline{A}}) = \sum_{k=1}^{n} a_{ik} C_{ik} \quad \text{or} \quad \det(\underline{\underline{A}}) = \sum_{k=1}^{n} a_{kj} C_{kj}.$$

Example: 3×3 matrix

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ -2 & -3 & 5 \end{pmatrix}.$$

We expand on the second column because it has a zero in it: the pattern of $(-1)^{i+j}$ will be $-$, $+$, $-$.

$$\begin{aligned} \det(\underline{\underline{A}}) &= -(2) \times \begin{vmatrix} -1 & 4 \\ -2 & 5 \end{vmatrix} + (0) \times \begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} - (-3) \times \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} \\ &= -2 \times [(-1)(5) - (-2)(4)] + 0 - (-3) \times [(1)(4) - (-1)(3)] \\ &= -2 \times [-5 + 8] + 3 \times [4 + 3] = -6 + 21 = 15. \end{aligned}$$

Properties

- $\det(\underline{\underline{A}}^{\top}) = \det(\underline{\underline{A}})$.
- For a **diagonal** matrix, or a (lower or upper) **triangular** matrix, the determinant is just the product of the diagonal elements.
- If we change $\underline{\underline{A}}$ by multiplying a whole row (or column) by k, then the determinant is multiplied by k. Thus, if the whole $n \times n$ matrix is multiplied by k, the determinant is multiplied by k^n.
- If we add (or subtract) a multiple of a row to another row, the determinant doesn't change.
- If we swap two rows (or two columns) the determinant changes by a factor of (-1).
- If an entire row or column is zero, the determinant is zero
- $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$.
- If one row or column is written as a sum, then the determinant can be written as the sum of two determinants:
$$\begin{vmatrix} a_{11} + \alpha & a_{12} & a_{13} \\ a_{21} + \beta & a_{22} & a_{23} \\ a_{31} + \gamma & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \alpha & a_{12} & a_{13} \\ \beta & a_{22} & a_{23} \\ \gamma & a_{32} & a_{33} \end{vmatrix}$$