Analytical Methods: Solutions 4

1. $\underline{\nabla} \cdot \underline{u} = 0$, $\underline{u} \cdot \underline{\nabla u} = -\underline{\nabla p} + \nabla^2 \underline{u}$: Blasius boundary layer.

   We scale $X = \varepsilon^a x$, $Y = \varepsilon^b y$, $U = \varepsilon^c u$ and $V = \varepsilon^d v$, and $P = \varepsilon^e p$ (where e is not the base of natural logarithms), and note that we are expecting $v \ll u$ so $d < c$. We obtain the conditions:

   $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Longrightarrow \qquad a - c = b - d$$

   which means that $b < a$. This means that we can omit the $\partial^2/\partial X^2$ terms from the Laplacian because we know that the $\partial^2/\partial Y^2$ terms dominate over them. Then the momentum equations give:

   $$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right) u = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \qquad \Longrightarrow$$

   $$\varepsilon^{a-2c} \left(U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} \right) = -\varepsilon^{a-e}\frac{\partial P}{\partial X} + \varepsilon^{2b-c}\frac{\partial^2 U}{\partial Y^2}$$

   with the two conditions

   $$e = 2c \qquad a = 2b + c.$$

   Finally the second momentum equation gives

   $$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right) v = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial y^2} \qquad \Longrightarrow$$

   $$\varepsilon^{3b} \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} \right) = -\varepsilon^{b-2c}\frac{\partial P}{\partial Y} + \varepsilon^{3b}\frac{\partial^2 V}{\partial Y^2}$$

   These terms can only all balance if $b = -c$, which would give scalings $a = b = -c = -d$, which violates our assumption that $d < c$. Instead we accept that the pressure term dominates this equation and the leading order equation is $\partial P/\partial Y = 0$.

   We return to the x-momentum equation and the mass conservation equation. We have the conditions

   $$e = 2c \qquad a = 2b + c \qquad a - c = b - d$$

   and we are solving the dominant equations

   $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}.$$

   Now the following quantities are invariant under our transformation:

   $$x^{-b/a}y \qquad x^{-c/a}u \qquad x^{-d/a}v \qquad x^{-e/a}p$$

   Using our conditions to eliminate c, d and e and setting $b/a = m$ these become

   $$x^{-m}y \qquad x^{-(1-2m)}u \qquad x^{m}v \qquad x^{-2(1-2m)}p$$

so we pose the following solution:

$$u = x^{1-2m}U(\xi) \qquad v = x^{-m}V(\xi) \qquad p = x^{2(1-2m)}P(\xi) \qquad \xi = x^{-m}y$$

However, recall we had $\partial p/\partial y = 0$ which means $P(\xi)$ must in fact be a constant. The outer boundary condition on u fixes $m = 1/2$ and then the similarity solution is of the form

$$u = U(\xi) \qquad v = x^{-1/2}V(\xi) \qquad p = P_0 \qquad \xi = x^{-1/2}y$$

and the full set of boundary conditions becomes

$$U(0) = V(0) = 0 \qquad U(\xi) \to 1, \quad V(\xi) \to 0 \ \text{ as } \xi \to \infty.$$

In order to reduce these to a single variable, we introduce a streamfunction ψ so that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$: this gives

$$\psi = x^{1/2}f(\xi) \ \text{ with } \ U(\xi) = f'(\xi), \quad V(\xi) = \frac{1}{2}[\xi f'(\xi) - f(\xi)].$$

The mass equation is then automatically satisfied and the x-momentum equation becomes

$$2f'''(\xi) + f(\xi)f''(\xi) = 0,$$

which may only be solved numerically. The boundary conditions which apply to it are

$$f(0) = f'(0) = 0 \qquad f'(\xi) \to 1 \ \text{ as } \ \xi \to \infty.$$

2. $J_\nu(\nu z) = \dfrac{1}{2\pi i}\displaystyle\int_{\infty - i\pi}^{\infty + i\pi} \exp\left[\nu z \sinh t - \nu t\right] \, \mathrm{d}t.$

Let us look at the behaviour of the function $f(t) = z\sinh t - t$. If we put $t = x + iy$ then

$$\mathrm{Real}\,(f(t)) = z \sinh x \cos y - x$$

which has deep valleys in positive x at $y = -\pi$ and $y = \pi$ and a high shoulder at $y = 0$. At the start and end points of our contour,

$$\mathrm{Real}\,(f(t)) = -z\sinh x - x \to -\infty.$$

The function only has stationary points at

$$z\cosh t - 1 = 0 \qquad \cosh t = \frac{1}{z} \qquad t = \pm\,\mathrm{arccosh}\left[\frac{1}{z}\right],$$

which means we can select a contour which comes up one of these valleys, crosses over at the positive saddle point and then descends the other valley. The strict steepest descent contour would keep $\mathrm{Imag}\,(f(t)) = 0$ throughout, and would be defined by

$$z \sin y \cosh x = y$$

but close to the point $x = t_0$, $y = 0$ we can approximate this contour by the quadratic

$$y^2 = 6z(x - t_0)\sinh t_0.$$

Now we can choose any contour which stays below the level of the saddle as it descends. One such choice is the contour consisting of two straight lines and our quadratic function:

$$
\begin{array}{lll}
C_1 & : \quad t = -s - i\pi & -\infty < s < -t_0 - \pi^2/6z\sinh t_0 \\
C_2 & : \quad t = t_0 + s^2/6z\sinh t_0 + is & -\pi < s < \pi \\
C_3 & : \quad t = s + i\pi & t_0 + \pi^2/6z\sinh t_0 < s < \infty
\end{array}
$$

The maximum contribution from C_1 and C_3 comes at the corner points where they meet C_2; however, the value there is exponentially lower than that along the contour C_2, which we will take as the leading component of our expansion:

$$J_\nu(\nu z) \sim \frac{1}{2\pi i}\int_{C_2} \exp\left[\nu z\sinh t - \nu t\right] \, \mathrm{d}t$$

Now along C_2, and close to its centre,

$$
\begin{aligned}
\mathrm{Real}\,(f(t)) &= z\sinh\left[t_0 + \frac{s^2}{6z\sinh t_0}\right]\cos s - t_0 - \frac{s^2}{6z\sinh t_0} \\
&\approx z\sinh t_0 - t_0 - z\sinh t_0\frac{s^2}{2} + O(s^4) \\
\mathrm{Imag}\,(f(t)) &= z\cosh\left[t_0 + \frac{s^2}{6z\sinh t_0}\right]\sin s - s \\
&\approx s^5\left\{\frac{1}{72z^2\sinh^2 t_0} - \frac{7}{360}\right\} + O(s^7)
\end{aligned}
$$

We can now calculate the leading terms of the integral:

$$
\begin{aligned}
J_\nu(\nu z) &\sim \frac{1}{2\pi i}\int_{C_2} \exp\left[\nu z\sinh t - \nu t\right] \, \mathrm{d}t \\
&\sim \frac{1}{2\pi i}\int \exp\left[\nu\left(z\sinh t_0\left[1 - \frac{s^2}{2}\right] - t_0\right)\right]\left(i + \frac{s}{3z\sinh t_0}\right) \, \mathrm{d}s
\end{aligned}
$$

The term $i + s/(3z\sinh t_0)$ comes from the change of variables in the integral; it is clear that the part with the s will result in an exact integral which will not contribute anything. Thus we have

$$
\begin{aligned}
J_\nu(\nu z) &\sim \frac{1}{2\pi}\int \exp\left[\nu\left(z\sinh t_0 - t_0 - z\sinh t_0\frac{s^2}{2}\right)\right] \, \mathrm{d}s \\
&\sim \frac{1}{2\pi}\exp\left[\nu\left(z\sinh t_0 - t_0\right)\right]\int \exp\left[\nu\left(-z\sinh t_0\frac{s^2}{2}\right)\right] \, \mathrm{d}s \\
&\sim \frac{1}{2\pi}\exp\left[\nu\left(z\sinh t_0 - t_0\right)\right]\left(\frac{2\pi}{\nu z\sinh t_0}\right)^{1/2}
\end{aligned}
$$

and substituting in the definition of t_0:

$$\cosh t_0 = \frac{1}{z} \qquad \sinh t_0 = \frac{(1-z^2)^{1/2}}{z}$$

we obtain

$$J_\nu(\nu z) \sim \left(\frac{1}{2\pi\nu(1-z^2)^{1/2}}\right)^{1/2}\exp\left[\nu\left((1-z^2)^{1/2} - \operatorname{arccosh}(1/z)\right)\right].$$