Analytical Methods: Solutions 3

1. $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{x^2}{(t+1)^2}\dfrac{\partial^2 u}{\partial x^2} = 0.$

   We try solutions of the form $X(x)T(t)$ to have

$$\frac{(t+1)^2 T''(t)}{T(t)} = \frac{x^2 X''(x)}{X(x)} = A$$

which gives the coupled ODEs

$$x^2 X''(x) = AX(x) \qquad (t+1)^2 T''(t) = AT(t).$$

The first of these has boundary conditions $X(1) = X(2) = 0$. It has solutions of the form x^m where

$$m^2 - m - A = 0 \qquad m = \frac{1 \pm \sqrt{1+4A}}{2}.$$

Putting $X(x) = ax^{m_1} + bx^{m_2}$, the two boundary conditions give

$$a + b = 0 \qquad a2^{m_1} + b2^{m_2} = 0$$

so we are constrained by

$$2^{m_1} = 2^{m_2}.$$

Note that $m_1 \neq m_2$ (since otherwise we would have $X(x) = 0$). The equation above can only be satisfied if m_1 and m_2 have equal real part: then given that the quadratic was real, they must be a complex conjugate pair. Say

$$m_1 = a + ib \qquad m_2 = a - ib$$

Then we need

$$2^{a+ib} = 2^{a-ib} \qquad 2^{ib} = 2^{-ib} \qquad \exp[ib\ln 2] = \exp[-ib\ln 2]$$

$$ib\ln 2 = -ib\ln 2 + 2n\pi i \qquad b = n\pi/\ln 2.$$

Returning to the definition of m, we have

$$a \pm \frac{in\pi}{\ln 2} = \frac{1 \pm \sqrt{1+4A}}{2} \qquad a = \frac{1}{2} \qquad A = -\frac{1}{4} - \left(\frac{n\pi}{\ln 2}\right)^2$$

and the solution for $X(x)$ is

$$X_n(x) = \alpha_n x^{1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$$

Now we return to the $T(t)$ equation, using our value for A:

$$(t+1)^2 T''(t) + \left[\frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2\right] T(t) = 0.$$

This has solutions of the form $T(t) = (t+1)^\lambda$ where

$$\lambda^2 - \lambda + \frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2 = 0 \qquad \lambda = \frac{1}{2} \pm i\frac{n\pi}{\ln 2}$$

so we have

$$T(t) = (t+1)^{1/2} \left\{ \beta \exp\left[\frac{in\pi}{\ln 2} \ln(t+1)\right] + \gamma \exp\left[-\frac{in\pi}{\ln 2} \ln(t+1)\right] \right\}.$$

The initial condition $u(x,0) = 0$ gives $T(0) = 0$ and thus $\beta + \gamma = 0$ and the full solution becomes:

$$u(x,t) = \sum_n \alpha_n x^{1/2} (t+1)^{1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) \sin\left(\frac{n\pi \ln(t+1)}{\ln 2}\right).$$

2. $\varepsilon \dfrac{\partial^2 u}{\partial x^2} + \varepsilon \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial y} = 0$

   (a) Trying $u = u_0 + \varepsilon u_1 + \cdots$ gives for the first two equations

   $$\begin{aligned} & & & & \partial u_0/\partial y &= 0 \\ \varepsilon \partial^2 u_0/\partial x^2 &+& \varepsilon \partial^2 u_0/\partial y^2 &+& \partial u_1/\partial y &= 0 \end{aligned}$$

   The leading-order solution is

   $$u_0 = f_0(x) \qquad u_0 = 1 - x^2$$

   and the next order equation becomes

   $$\partial u_1/\partial y = 2 \qquad u_1 = 2y + f_1(x).$$

   (b) Scaling $y = a + \delta Y$ will not affect the first term; to balance the second and third we need $\varepsilon \delta^{-2} = \delta^{-1}$ so $\delta = \varepsilon$. The new governing equation is

   $$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial Y^2} + \frac{\partial u}{\partial Y} = 0$$

   so if we pose

   $$f \sim f_0 + \varepsilon f_1 + \cdots$$

   then at both leading order and next order we have

   $$\partial^2 f_i/\partial Y^2 + \partial f_i/\partial Y = 0 \qquad f_i(x,Y) = A_i(x) \exp[-Y] + B_i(x).$$

   The boundary condition at $y = 0$ transforms to

   $$\frac{\partial u}{\partial Y}(x,0) + u(x,0) = 0 \qquad B_i(x) = 0.$$

   The solution is $f \sim A_0(x) e^{-Y} + \varepsilon A_1(x) e^{-Y} + \cdots$.

   (c) Return to the whole equation, and substitute $u = X(x)Y(y)$:

   $$\varepsilon \frac{X''(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} - \varepsilon \frac{Y''(y)}{Y(y)} = A$$

   The boundary conditions $u(-1,y) = u(1,y) = 0$ convert to $X(-1) = X(1) = 0$.
   If A is positive, $A = \varepsilon \lambda^2$ and

   $$X''(x) = \lambda^2 X(x) \qquad X(x) = a \exp[\lambda x] + b \exp[-\lambda x]$$

The boundary conditions cannot both be satisfied so we discard this solution.
If $A = 0$ we have $X(x) = \alpha x + \beta$ which cannot satisfy the x-boundary conditions.
Finally, if A is negative, $A = -\varepsilon\mu^2$ and

$$X''(x) = -\mu^2 X(x) \qquad X(x) = a\cos[\mu x] + b\sin[\mu x]$$

The boundary conditions fix $b = 0$, $\mu = (2n+1)\pi/2$.
Then we have

$$\varepsilon Y''(y) + Y'(y) - \varepsilon\mu^2 Y(y) = 0$$

which has solutions $Y = e^{my}$ if

$$m = \frac{-1 \pm \sqrt{1 + 4\varepsilon^2\mu^2}}{2\varepsilon} = \frac{-1 \pm \sqrt{1 + (2n+1)^2\pi^2\varepsilon^2}}{2\varepsilon}$$

The solution satisfying the boundary conditions at $x = \pm 1$ is

$$u = \sum_n a_n \cos\left[\frac{(2n+1)\pi x}{2}\right] (c_n \exp[m_1 y] + d_n \exp[m_2 y])$$

with

$$m_1 = \frac{-1 + \sqrt{1 + (2n+1)^2\pi^2\varepsilon^2}}{2\varepsilon} \qquad m_2 = \frac{-1 - \sqrt{1 + (2n+1)^2\pi^2\varepsilon^2}}{2\varepsilon}.$$

(d) When ε is small, we can expand the forms of m_1 and m_2:

$$m_1 \approx \frac{(2n+1)^2\pi^2\varepsilon}{4} \qquad m_2 \approx \frac{-1}{\varepsilon}.$$

So each of our Y functions consists of a slow exponential which grows with increasing y and a fast exponential which decreases with increasing y. The fast exponential is the scaled solution of (b); away from a small region near $y = 0$, it is essentially zero. The slow exponential contains the rest of the information in the domain.
Away from the region of small y, we have

$$u \approx \sum_n a_n c_n \cos\left[\frac{(2n+1)\pi x}{2}\right]$$

and if we apply the boundary condition at $y = 1$ using a Fourier series, we can see that we have $u \approx 1 - x^2$ everywhere except close to $y = 0$. This is the solution we found in (a).

3. The image of $|z - 1| \leq 1$ under $w = 1/z$.

The boundary of the domain may be parametrised as

$$|z - 1| = 1 \qquad z = 1 + e^{i\theta} \qquad 0 \leq \theta < 2\pi$$

which transforms to

$$w = \frac{1}{1 + \cos\theta + i\sin\theta} = \frac{1 + \cos\theta - i\sin\theta}{(1 + \cos\theta)^2 + \sin^2\theta} = \frac{1 + \cos\theta - i\sin\theta}{2 + 2\cos\theta}$$

$$w = \frac{1}{2} - i\frac{\sin\theta}{2 + 2\cos\theta}$$

The real part of w is always $1/2$; the imaginary part spans the whole line from negative infinity at $\theta = \pi$ to positive infinity at $\theta = -\pi$.

To complete the mapping we simply need to know which side of the boundary our domain lies. The point $z = 1$ is in the original domain: therefore the point $w = 1$ is in the image domain, which is therefore given by

$$\text{Real}\,(w) \geq \frac{1}{2}.$$

4. The image of $-\pi/2 < x < \pi/2$, $0 < y < 1$ under $w = \sin z$.

   We look at each boundary in turn, writing $w = \eta + i\xi$ where necessary.

   Bottom edge $y = 0$: $w = \sin x$, $-\pi/2 < x < \pi/2$.

   $$-1 < \eta < 1, \quad \xi = 0.$$

   Top edge $y = 1$: $w = \sin(x + i) = \cosh 1 \sin x + i \sinh 1 \cos x$.

   $$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1, \quad \xi > 0.$$

   Left edge $x = -\pi/2$: $w = \sin(-\pi/2 + iy) = -\cosh y$, $0 < y < 1$.

   $$-\cosh 1 < \eta < -1 \qquad \xi = 0.$$

   Right edge $x = \pi/2$: $w = \sin(\pi/2 + iy) = \cosh y$, $0 < y < 1$.

   $$1 < \eta < \cosh 1 \qquad \xi = 0.$$

   The top edge is half an ellipse; the other three form the straight line $-\cosh 1 < \eta < \cosh 1$, $\xi = 0$. We now need to check whether the interior or exterior of the half-ellipse is our image. Take a point from the interior of the rectangle – say, $z = i/2$. Then $w = \sin i/2 = i \sinh(1/2)$, which is inside our half ellipse. The image domain is

   $$w = \eta + i\xi \qquad \frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} \leq 1, \quad \xi > 0.$$

5. The image of $-\pi/4 < x < \pi/4$, $-1 < y < 1$ under $w = \sin z$.

   Again, we find the image of each edge in turn, putting $z = x + iy$, $w = \eta + i\xi$.

   Bottom edge $y = -1$: $w = \sin(x - i) = \cosh 1 \sin x - i \sinh 1 \cos x$.

   $$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1 \qquad -\frac{\cosh 1}{\sqrt{2}} < \eta < \frac{\cosh 1}{\sqrt{2}} \qquad \xi < -\frac{\sinh 1}{\sqrt{2}}.$$

   Top edge $y = 1$: As in 4, but with a reduced range of x:

   $$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1 \qquad -\frac{\cosh 1}{\sqrt{2}} < \eta < \frac{\cosh 1}{\sqrt{2}} \qquad \xi > \frac{\sinh 1}{\sqrt{2}}.$$

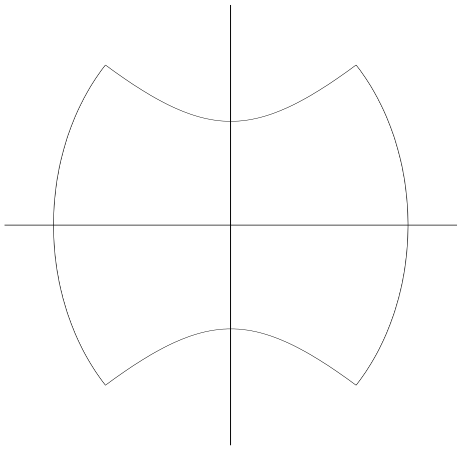
Left edge $x = -\pi/4$**:** $w = \sin(iy - \pi/4) = (-\cosh y + i\sinh y)/\sqrt{2}$.

$$\eta^2 - \xi^2 = \frac{1}{2} \qquad -\frac{\cosh 1}{\sqrt{2}} < \eta < -\frac{1}{\sqrt{2}} \qquad -\frac{\sinh 1}{\sqrt{2}} < \xi < \frac{\sinh 1}{\sqrt{2}}.$$

Right edge $x = \pi/4$**:** $w = \sin(iy + \pi/4) = (\cosh y + i\sinh y)/\sqrt{2}$.

$$\eta^2 - \xi^2 = \frac{1}{2} \qquad \frac{1}{\sqrt{2}} < \eta < \frac{\cosh 1}{\sqrt{2}} \qquad -\frac{\sinh 1}{\sqrt{2}} < \xi < \frac{\sinh 1}{\sqrt{2}}.$$

This curvilinear rectangle looks like:



and is bounded by the hyperbola $\eta^2 - \xi^2 = 1/2$ and the ellipse

$$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1$$

6. $\nabla^2 u = 0$ in the domain $1 < r < e^\alpha$, $0 < \alpha < \pi$ with boundary conditions $\partial u/\partial r(1, \theta) = 0$, $\partial u/\partial r(e^\alpha, \theta) = \sin\theta$, $u(r, 0) = u(r, \pi) = 0$.

   (a) The geometry of the domain suggests polar coordinates, in which

   $$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

   and the three types of separable solution are

   $$u = (a\cos[\lambda\theta] + b\sin[\lambda\theta])(cr^\lambda + dr^{-\lambda})$$
   $$u = (A\exp[\mu\theta] + B\exp[-\mu\theta])(C\cos[\mu\ln r] + D\sin[\mu\ln r])$$
   $$u = (\alpha + \beta\ln r)(\gamma + \delta\theta).$$

   The boundary conditions $u(r, 0) = u(r, \pi) = 0$ impose $a = A = B = \gamma = \delta = 0$ and $\lambda = n$ so

   $$u = \sum_n \sin[n\theta](c_n r^n + d_n r^{-n}).$$

Then
$$\frac{\partial u}{\partial r} = \sum_n \sin[n\theta](ncr^{n-1} - ndr^{-(n+1)})$$
and the boundary conditions $\partial u/\partial r(1,\theta) = 0$ and $\partial u/\partial r(e^\alpha,\theta) = \sin\theta$ give $n = 1$ and finally
$$u(r,\theta) = \frac{(r + r^{-1})\sin\theta}{(1 - e^{-2\alpha})}.$$

(b) Under $w = \ln z$ with $w = \eta + i\xi$, the region $0 < \theta < \pi$, $1 < r < e^\alpha$ becomes $0 < \eta < \alpha$, $0 < \xi < \pi$. The three zero boundary conditions become
$$u(\eta,0) = 0 \qquad u(\eta,\pi) = 0 \qquad \frac{\partial u}{\partial \eta}(0,\xi) = 0$$
and since $|\mathrm{d}w/\mathrm{d}z| = |1/z| = e^{-\alpha}$ on $r = e^\alpha$, the final boundary condition becomes
$$\frac{\partial u}{\partial \eta}(\alpha,\xi) = e^\alpha \sin\xi.$$
It is clear that the solution is
$$u = \sin\xi(a\cosh\eta + b\sinh\eta) \qquad \frac{\partial u}{\partial \eta} = \sin\xi(a\sinh\eta + b\cosh\eta)$$
$$u = \frac{e^\alpha \cosh\eta \sin\xi}{\sinh\alpha} = \frac{2\cosh\eta\sin\xi}{(1 - e^{-2\alpha})}$$
Now if $\ln z = \eta + i\xi$ then $\eta = \ln r$ and $\xi = \theta$ so our solution is
$$u(z) = \frac{2\cosh[\ln r]\sin\theta}{(1 - e^{-2\alpha})} = \frac{(r + r^{-1})\sin\theta}{(1 - e^{-2\alpha})}.$$
Note the analytic function of which u is the real part is
$$f = \frac{2\sin(-iw)}{(1 - e^{-2\alpha})} = \frac{2\sin(-i\ln z)}{(1 - e^{-2\alpha})}.$$

7. The two distinguished stretches are $\delta = 1$ and $\delta = \varepsilon$. $\delta = 1$ gives us a regular expansion:
$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$
$$\frac{\mathrm{d}f_0}{\mathrm{d}x} = \cos x \qquad \frac{\mathrm{d}f_1}{\mathrm{d}x} = -\frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} = \sin x \qquad \frac{\mathrm{d}f_2}{\mathrm{d}x} = -\frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} = -\cos x$$
$$f = c_0 + \sin x + \varepsilon[c_1 - \cos x] + \varepsilon^2[c_2 - \sin x] + \cdots$$
in which the boundary condition at $x = \pi$ gives:
$$1 = c_0 + \varepsilon[c_1 + 1] + \varepsilon^2[c_2] + \cdots$$
$$f(x) = 1 + \sin x - \varepsilon[1 + \cos x] - \varepsilon^2 \sin x + \cdots$$
For the stretch $\delta = \varepsilon$, if $z = x/\varepsilon$ then
$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}z} = \varepsilon\cos\varepsilon z = \varepsilon(1 - \varepsilon^2 z^2/2 + \cdots)$$

$$f = F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \cdots$$

$$\frac{\mathrm{d}^2F_0}{\mathrm{d}z^2} + \frac{\mathrm{d}F_0}{\mathrm{d}z} = 0 \qquad \frac{\mathrm{d}^2F_1}{\mathrm{d}z^2} + \frac{\mathrm{d}F_1}{\mathrm{d}z} = 1 \qquad \frac{\mathrm{d}^2F_2}{\mathrm{d}z^2} + \frac{\mathrm{d}F_2}{\mathrm{d}z} = 0$$

$$f(z) = a_0 + b_0e^{-z} + \varepsilon[a_1 + b_1e^{-z} + z] + \varepsilon^2[a_2 + b_2e^{-z}] + \cdots$$

which becomes (using the BC at $z = 0$):

$$f(z) = a_0 - a_0e^{-z} + \varepsilon[a_1 - a_1e^{-z} + z] + \varepsilon^2[a_2 - a_2e^{-z}] + \cdots$$

Now we match, using an intermediate variable $x = \varepsilon^\alpha\xi$. The expansion of the outer solution is

$$f(x) = 1 + \varepsilon^\alpha\xi - 2\varepsilon - \varepsilon^{3\alpha}\xi^3/6 + \varepsilon^{1+2\alpha}\xi^2/2 - \varepsilon^{2+\alpha}\xi + O(\varepsilon^3, \varepsilon^{5\alpha})$$

The inner expansion, with $z = \varepsilon^{\alpha-1}\xi$, becomes

$$f(z) = a_0 + \varepsilon^\alpha\xi + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots$$

Matching the two gives $a_0 = 1$, $a_1 = -2$ and $a_2 = 0$, so the matched inner form is

$$f(z) = 1 - e^{-z} + \varepsilon[2e^{-z} - 2 + z] + O(\varepsilon^3).$$

8. $(1 + \varepsilon)x^2y' = \varepsilon((1 - \varepsilon)xy^2 - (1 + \varepsilon)x + y^3 + 2\varepsilon y^2)$ with $y(1) = 1$.

Outer: set $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ to have (rows being order 1, ε, ε^2):

$$\begin{array}{lllllllllll}
x^2y_0' & & & = & 0 & & & & & & \\
x^2y_1' & + & x^2y_0' & = & xy_0^2 & & & - & x & + & y_0^3 \\
x^2y_2' & + & x^2y_1' & = & 2xy_0y_1 & - & xy_0^2 & - & x & + & 3y_0^2y_1 & + & 2y_0^2
\end{array}$$

Order 1 At leading order, $y_0' = 0$ gives $y_0 = a_0$ and, using the boundary condition, $y_0 = 1$.

Order ε The equation becomes $x^2y_1' = 1$ so $y_1 = a_1 - x^{-1}$. The boundary condition fixes $y_1 = 1 - x^{-1}$.

Order ε^2 The equation is $x^2y_2' = 2 - 3x^{-1}$ so $y_2 = a_2 - 2x^{-1} + (3/2)x^{-2}$. With the boundary condition we have $y_2 = 1/2 - 2x^{-1} + (3/2)x^{-2}$.

The outer expansion is

$$y \sim 1 + \varepsilon\left(1 - \frac{1}{x}\right) + \varepsilon^2\left(\frac{1}{2} - \frac{2}{x} + \frac{3}{2x^2}\right) + \cdots$$

which ceases to be uniformly asymptotic when $x \sim \varepsilon$.

We rescale (noting that y is still order 1) by putting $x = \varepsilon z$ and the original equation becomes:

$$(1 + \varepsilon)z^2y' = \varepsilon(1 - \varepsilon)zy^2 - \varepsilon(1 + \varepsilon)z + y^3 + 2\varepsilon y^2$$

In the inner, we pose $y = f_0 + \varepsilon f_1 + \cdots$ to have

$$\begin{array}{lllllllllll}
z^2f_0' & & & = & & & & & f_0^3 & & \\
\varepsilon z^2f_1' & + & \varepsilon z^2f_0' & = & \varepsilon zf_0^2 & - & \varepsilon z & + & 3\varepsilon f_0^2f_1 & + & 2\varepsilon f_0^2
\end{array}$$

Order 1: $z^2 f_0' = f_0^3$ has solution $f_0 = (A_0 + 2/z)^{-1/2}$.

Matching: We use an intermediate variable $\eta = \varepsilon^{-\alpha} x = \varepsilon^{1-\alpha} z$. The outer becomes

$$y \sim 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{3}{2\eta^2} + \varepsilon - \varepsilon^{2-\alpha}\frac{2}{\eta} + \frac{\varepsilon^2}{2} + \cdots$$

and the inner,

$$y \sim A_0^{-1/2}\left(1 - \varepsilon^{1-\alpha}\frac{1}{A_0\eta} + \varepsilon^{2-2\alpha}\frac{3}{2A_0^2\eta^2} + \cdots\right)$$

which matches the first three terms if we set $A_0 = 1$. So we have $f_0 = (1 + 2/z)^{-1/2}$.

Order ε: The equation becomes $z^2 f_1' - 3(1+2/z)^{-1} f_1 = -(1+2/z)^{-3/2}$ so (using the integrating factor $(1+2/z)^{3/2}$) we have

$$f_1 = \left(A_1 + \frac{1}{z}\right)\left(1 + \frac{2}{z}\right)^{-3/2}$$

Matching The outer is unchanged from before: the inner now becomes

$$y \sim 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{3}{2\eta^2} + \varepsilon A_1 + \varepsilon^{2-\alpha}\left(\frac{1-3A_1}{\eta}\right) + \cdots$$

which matches the next two terms of the outer if we set $A_1 = 1$.

The inner expansion is

$$y \sim \left(1 + \frac{2}{z}\right)^{-1/2} + \varepsilon\left(1 + \frac{1}{z}\right)\left(1 + \frac{2}{z}\right)^{-3/2} + \cdots$$