# Analytical Methods: Solutions 2

1. $\varepsilon x^3 + x^2 + (2-\varepsilon)x + 1 = 0$.

   Scale $x \sim \delta$ and note that the $\varepsilon x$ term is always smaller than the $2x$ term. Further, if $x^2 \gg 1$ then $x^2 \gg 2x$; if $x^2 \ll 1$ then $2x \ll 1$ so the $2x$ term only dominates if two other terms balance. We are now comparing the following terms:

   $$[\mathbf{A}]\ \ \varepsilon\delta^3 \qquad [\mathbf{B}]\ \ \delta^2 \qquad [\mathbf{C}]\ \ 1.$$

   For small $\delta$, **[C]** dominates. **[B]** catches up first at $\delta = 1$. Then **[A]** catches up with **[B]** when $\varepsilon\delta^3 = \delta^2$, $\delta = \varepsilon^{-1}$. The distinguished scalings are $x \sim 1$ and $x \sim \varepsilon^{-1}$.

   We solve first for the regular root(s): $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$
   Substiting in gives

   $$\begin{array}{ccccccccc} & & x_0^2 & + & 2x_0 & & & + \ 1 & = & 0 \\ \varepsilon x_0^3 & + & 2\varepsilon x_0 x_1 & + & 2\varepsilon x_1 & - & \varepsilon x_0 & & = & 0 \\ 3\varepsilon^2 x_0^2 x_1 & + & 2\varepsilon^2 x_0 x_2 + 2\varepsilon^2 x_1^2 & + & 2\varepsilon^2 x_2 & - & \varepsilon^2 x_1 & & = & 0 \end{array}$$

   The leading order term gives $x_0 = -1$. At order $\varepsilon$ we have

   $$-1 - 2x_1 + 2x_1 + 1 = 0$$

   which is automatically satisfied. At order $\varepsilon^2$ we obtain

   $$3x_1 - 2x_2 + 2x_1^2 + 2x_2 - x_1 = 0 \qquad 2x_1(1 + x_1) = 0$$

   which is satisfied either by $x_1 = -1$ or $x_1 = 0$. In fact $x = -1$ is an exact root:

   $$\varepsilon x^3 + x^2 + (2-\varepsilon)x + 1 = (x+1)(\varepsilon x^2 + (1-\varepsilon)x + 1)$$

   so no further terms are available for the other root.

   Looking for the singular root, we pose: $x = \varepsilon^{-1}x_{-1} + x_0 + \cdots$
   and substituting gives

   $$\begin{array}{cccccc} \varepsilon^{-2}x_{-1}^3 & + & \varepsilon^{-2}x_{-1}^2 & & & = & 0 \\ 3\varepsilon^{-1}x_{-1}^2 x_0 & + & 2\varepsilon^{-1}x_{-1}x_0 & + & 2\varepsilon^{-1}x_{-1} & = & 0 \end{array}$$

   At order $\varepsilon^{-2}$ we obtain $x_{-1} = -1$ (recall the leading $x$ term is strictly order 1 so 0 is not a valid solution); at order $\varepsilon^{-1}$ we get $x_0 = 2$. Thus the three roots are $x = -1 - 2\varepsilon + O(\varepsilon^2)$; $x = -1$; and $x = -\varepsilon^{-1} + 2 + O(\varepsilon)$.

2. $\varepsilon x^4 - x^2 - x + 2 = 0$.

   First we look for scalings. One of $x^2$ or 2 is always at least as large as $x$ so we only consider

   $$[\mathbf{A}]\ \ \varepsilon\delta^4 \qquad [\mathbf{B}]\ \ \delta^2 \qquad [\mathbf{C}]\ \ 1$$

   At small $\delta$ **[C]** is largest, and it is equalled first by **[B]** when $\delta = 1$. For larger $\delta$, **[A]** reaches **[B]** when $\varepsilon\delta^4 = \delta^2$ i.e. $\delta = \varepsilon^{-1/2}$.

Look at the regular root(s) first (and assume a regular expansion):

$$x = x_0 + \varepsilon x_1 + \cdots$$

gives

$$\begin{array}{ccccccccc} & - & x_0^2 & - & x_0 & + & 2 & = & 0 \\ \varepsilon x_0^4 & - & 2\varepsilon x_0 x_1 & - & \varepsilon x_1 & & & = & 0 \end{array}$$

At order 1 we have

$$x_0^2 + x_0 - 2 = 0 \qquad (x_0+2)(x_0-1) = 0 \qquad x_0 = 1 \text{ or } x_0 = -2.$$

If $x_0 = 1$ the next order gives $1 - 3x_1 = 0$ so $x \sim 1 + \varepsilon/3$.

If $x_0 = -2$ the next order gives $16 + 3x_1 = 0$ so $x \sim -2 - 16\varepsilon/3$.

Now we move on to the singular roots and the scaling suggests an expansion in $\varepsilon^{1/2}$:

$$x = \varepsilon^{-1/2}x_0 + x_1 + \varepsilon^{1/2}x_2 + \cdots$$

Substituting gives

$$\begin{array}{ccccccc} \varepsilon^{-1}x_0^4 & - & \varepsilon^{-1}x_0^2 & & & = & 0 \\ 4\varepsilon^{-1/2}x_0^3 x_1 & - & 2\varepsilon^{-1/2}x_0 x_1 & - & \varepsilon^{-1/2}x_0 & = & 0 \end{array}$$

At leading order we have

$$x_0^4 - x_0^2 = 0 \qquad x_0^2(x_0+1)(x_0-1) = 0 \qquad x_0 = 1 \text{ or } x_0 = -1.$$

If $x_0 = 1$ then the next order gives $2x_1 - 1 = 0$ so $x \sim \varepsilon^{-1/2} + 1/2$.

If $x_0 = -1$ then the next order gives $-2x_1 + 1 = 0$ so $x \sim -\varepsilon^{-1/2} + 1/2$.

3. $xe^{-x} = \varepsilon$. Define $f(x) = xe^{-x}$. This function is positive for $x > 0$; zero at both $x = 0$ and $x \to \infty$; and $f(1) = e^{-1} \gg \varepsilon$ so we expect two roots, one in $0 < x < 1$ and the other in $1 < x < \infty$.

   Let us look first for the root near $x = 0$. We can expand the exponential:

$$\varepsilon = x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right) = x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots$$

   It is clear that the leading scaling is $x \sim \varepsilon$. Looking for the next order, set $x = \varepsilon + \delta_1 x_1 + \cdots$:

$$\varepsilon = \varepsilon + \delta_1 x_1 - \varepsilon^2 + O(\varepsilon^3, \varepsilon^2\delta_1)$$

   which gives $\delta_1 x_1 = \varepsilon^2$ and the beginning of the expansion is

$$x \sim \varepsilon + \varepsilon^2 + \cdots$$

   The root in $x > 1$ depends more strongly on the exponential than on the $x$ term, so we try a logarithmic scaling. Let us try the values of $f(x)$ when $x = x_0 \ln(1/\varepsilon)$:

   - If $x_0 = 1$ then $f(x) = \varepsilon \ln(1/\varepsilon) \gg \varepsilon$.
   - If $x_0 = 2$ then $f(x) = 2\varepsilon^2 \ln(1/\varepsilon) \ll \varepsilon$.

These two points bracket the root so we know the scaling is correct. We begin our expansion

$$x = x_0 \ln(1/\varepsilon) + \delta_1 x_1 + \cdots$$

and substitute it in, using $L_1 = \ln(1/\varepsilon)$, to obtain:

$$\varepsilon = \varepsilon^{x_0}(x_0 L_1 + \delta_1 x_1 + \cdots) \exp[\delta_1 x_1 + \cdots] = \varepsilon^{x_0} x_0 L_1 \exp[\delta_1 x_1] + \cdots$$

To match the powers of $\varepsilon$ we need $x_0 = 0$; then to make the logarithm terms work we need

$$\delta_1 x_1 = -L_2 \qquad \delta_1 = L_2, \quad x_1 = -1.$$

in which we have used $L_2 = \ln L_1$.

The beginning of the expansion is

$$x \sim \ln(1/\varepsilon) - \ln(\ln(1/\varepsilon)) + \cdots$$

4. $u = \dfrac{1}{2c} \displaystyle\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x', t')\,\mathrm{d}x'\,\mathrm{d}t'.$

The easy part is the derivatives wrt $x$: for ease, split the integral in two:

$$u = \frac{1}{2c} \int_0^t \int_0^{x+c(t-t')} F(x', t')\,\mathrm{d}x'\,\mathrm{d}t' - \frac{1}{2c} \int_0^t \int_0^{x-c(t-t')} F(x', t')\,\mathrm{d}x'\,\mathrm{d}t'$$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{2c} \left( \int_0^t F(x + c(t-t'), t')\,\mathrm{d}t' - \int_0^t F(x - c(t-t'), t')\,\mathrm{d}t' \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2c} \left( \int_0^t F_x(x + c(t-t'), t')\,\mathrm{d}t' - \int_0^t F_x(x - c(t-t'), t')\,\mathrm{d}t' \right)$$

Partial derivatives wrt $t$ must be taken with more care, as both inner and outer integrals have limits which depend on $t$:

$$\frac{\partial u}{\partial t} = \left( \frac{1}{2c} \int_0^{x+c(t-t)} F(x', t)\,\mathrm{d}x' - \frac{1}{2c} \int_0^{x-c(t-t)} F(x', t)\,\mathrm{d}x' \right)$$
$$+ \left( \frac{1}{2c} \int_0^t cF(x + c(t-t'), t')\,\mathrm{d}t' - \frac{1}{2c} \int_0^t (-c)F(x - c(t-t'), t')\,\mathrm{d}t' \right)$$

The first two terms here cancel and we are left with

$$\frac{\partial u}{\partial t} = \frac{1}{2} \int_0^t F(x + c(t-t'), t') + F(x - c(t-t'), t')\,\mathrm{d}t'$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} (F(x + c(t-t), t) + F(x - c(t-t), t))$$
$$+ \frac{1}{2} \int_0^t cF_x(x + c(t-t'), t') - cF_x(x - c(t-t'), t')\,\mathrm{d}t'$$

Putting these together gives the required result:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x, t).$$

5. $\partial^2 u/\partial t^2 - e^{2x}\partial^2 u/\partial x^2$. Here $c(x) = e^x$ so the characteristics are given by

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \pm e^x \qquad t = \pm e^x + \alpha$$

Through the point $x = 0$ , $t = 1$, the two equations become

$$t = e^x \quad \text{and} \quad t = 2 - e^x.$$

6. $\dfrac{\partial^2 f}{\partial t^2} - \dfrac{\partial^2 f}{\partial x^2} - \varepsilon \cos x f = x$ with $f(x,0) = \dfrac{\partial f}{\partial t}(x,0) = 0$.
Set $f = f_0 + \varepsilon f_1 + \cdots$.

$$\begin{array}{ccccccc} \dfrac{\partial^2 f_0}{\partial t^2} & - & \dfrac{\partial^2 f_0}{\partial x^2} & & & = & x \\ \varepsilon\dfrac{\partial^2 f_1}{\partial t^2} & - & \varepsilon\dfrac{\partial^2 f_1}{\partial x^2} & - & \varepsilon \cos x\, f_0 & = & 0 \end{array}$$

It is useful to note that the solution of the inhomogeneous wave equation with $c = 1$:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = F(x,t)$$

is

$$f(x,t) = p(x+t) + q(x-t) + \frac{1}{2}\int_0^t \int_{x-t+t'}^{x+t-t'} F(x',t')\,\mathrm{d}x'\,\mathrm{d}t'$$

and applying the boundary conditions $f(x,0) = \partial f/\partial t(x,0) = 0$ to this form gives

$$f(x,t) = \frac{1}{2}\int_0^t \int_{x-t+t'}^{x+t-t'} F(x',t')\,\mathrm{d}x'\,\mathrm{d}t'.$$

**Order 1.** The inhomogeneous wave equation

$$\frac{\partial^2 f_0}{\partial t^2} - \frac{\partial^2 f_0}{\partial x^2} = x \qquad f_0(x,0) = \frac{\partial f_0}{\partial t}(x,0) = 0$$

has the solution

$$\begin{aligned} f_0(x,t) &= \frac{1}{2}\int_0^t \int_{x-t+t'}^{x+t-t'} x'\,\mathrm{d}x'\,\mathrm{d}t' = \frac{1}{2}\int_0^t \left[\frac{1}{2}x'^2\right]_{x-t+t'}^{x+t-t'} \mathrm{d}t' \\ &= \int_0^t x(t-t')\,\mathrm{d}t' = \left[x(tt' - \tfrac{1}{2}t'^2)\right]_0^t = \tfrac{1}{2}xt^2. \end{aligned}$$

$$f_0(x,t) = \tfrac{1}{2}xt^2.$$

**Order $\varepsilon$.** We are solving

$$\frac{\partial^2 f_1}{\partial t^2} - \frac{\partial^2 f_1}{\partial x^2} = f_0 \cos x = \tfrac{1}{2}x \cos x t^2$$

with the same zero boundary conditions as before. The solution is

$$f_1(x,t) = \frac{1}{4}\int_0^t \int_{x-t+t'}^{x+t-t'} x' \cos x' t'^2\,\mathrm{d}x'\,\mathrm{d}t'$$

which becomes, after tedious but straightforward work,

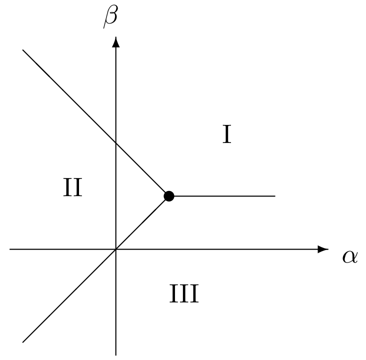
$$f_1(x,t) = \tfrac{1}{2}t^2(x\cos x - 2\sin x) - x\cos x + 4\sin x$$
$$+\tfrac{1}{2}(x+t)\cos(x+t) + \tfrac{1}{2}(x-t)\cos(x-t) - 2\sin(x+t) - 2\sin(x-t).$$

7. $\varepsilon\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\dfrac{\mathrm{d}f}{\mathrm{d}x} - f = 0.$

   (a) Scaling the equation with $f = \varepsilon^\alpha F$ and $x = a + \varepsilon^\beta z$ and dividing by $\varepsilon^\alpha$ gives
   $$\varepsilon^{1-2\beta}\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + \varepsilon^{\alpha-\beta}F\frac{\mathrm{d}F}{\mathrm{d}z} - F = 0$$
   The balances are:

   - I and II: $1-2\beta = \alpha-\beta$ so $\alpha+\beta = 1$. These two terms dominate if $\alpha - \beta < 0$.
   - I and III: $1 - 2\beta = 0$ so $\beta = 1/2$. These terms dominate if $\alpha - \beta > 0$.
   - II and III: $\alpha-\beta = 0$ so $\alpha = \beta$. These terms dominate if $1-2\beta > 0$ so $\beta < 1/2$.

   The scalings in the $\alpha$–$\beta$ plane are:




   (b) The critical scaling at which all terms balance is $\alpha = \beta = 1/2$.

   (c) If we fix $\alpha = 0$ then the two possible balances are between terms I and II, in which case $\beta = 1$, and terms II and III, in which case $\beta = 0$.

   - Setting $\beta = 0$ gives a regular expansion: the leading-order equation is $f_0(f_0' - 1) = 0$ and since $f_0$ cannot be zero (strictly order 1) we have $f_0' = 1$ and the solution is $f_0 = a_0 + x$.
   - Setting $\beta = 1$ we put $x = a + \varepsilon z$ and the governing equation becomes
   $$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f\frac{\mathrm{d}f}{\mathrm{d}z} - \varepsilon f = 0.$$
   The leading-order equation is
   $$F_0'' + F_0F_0' = 0 \qquad F_0' + \tfrac{1}{2}F_0^2 = A_0 \qquad F_0' = A_0 - \tfrac{1}{2}F_0^2$$

which has three possible solutions according to the sign of $A_0$: if $A_0 = -2k^2$ then

$$-\int \mathrm{d}z = \int \frac{2\,\mathrm{d}F_0}{4k^2 + F_0^2} = \frac{\arctan(F_0/2k)}{k}$$

$$F_0 = -2k \tan[k(z + B_0)];$$

if $A_0 = 2k^2$ then

$$\int 2k\ \mathrm{d}z = \int \frac{4k\,\mathrm{d}F_0}{4k^2 - F_0^2} = \int \frac{4k\,\mathrm{d}F_0}{(2k + F_0)(2k - F_0)}$$

$$2k(z + B_0) = \int \left( \frac{1}{(2k + F_0)} + \frac{1}{(2k - F_0)} \right) \mathrm{d}F_0 = \ln \left| \frac{2k + F_0}{2k - F_0} \right|$$

which in turn has three possible solutions:

$$F_0 = \pm 2k \quad F_0 = 2k \tanh[k(z + B_0)] \quad F_0 = 2k \coth[k(z + B_0)];$$

and finally, if $A_0 = 0$ then

$$\int \mathrm{d}z = \int \frac{-2\,\mathrm{d}F_0}{F_0^2} = \frac{2}{F_0} - B_0 \qquad F_0 = \frac{2}{z + B_0}.$$

8. $\varepsilon^3 \dfrac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \varepsilon \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \dfrac{\mathrm{d}f}{\mathrm{d}x} + f = 0.$
Putting $x = a + \delta X$, the terms scale as

$$[\mathbf{A}]\ \varepsilon^3\delta^{-3} \qquad [\mathbf{B}]\ \varepsilon\delta^{-2} \qquad [\mathbf{C}]\ \delta^{-1} \qquad [\mathbf{D}]\ 1$$

For very small $\delta$, **[A]** dominates. It is first caught by **[B]** at $\delta = \varepsilon^2$. **[B]** is overtaken by **[C]** at $\delta = \varepsilon$ and finally **[C]** is balanced by **[D]** at $\delta = 1$. The three distinguished stretches are $\delta = 1$, $\delta = \varepsilon$ and $\delta = \varepsilon^2$.

Using $\delta = 1$, the leading-order equation is $f' + f = 0$ giving $f = be^{-x}$.
Using $x = a + \varepsilon y$, the leading equation is $f'' + f' = 0$, so $f = be^{-y} + c$.
Using $x = a + \varepsilon^2 z$, the leading equation is $f''' + f'' = 0$, so $f = be^{-z} + cz + d$.

9. (a) The discriminant is $B^2 - 4AC = t^2 - 4x$ so the operator is hyperbolic in $t^2 > 4x$, parabolic on the parabola $t^2 = 4x$ and elliptic where $t^2 < 4x$.
   (b) The discriminant is $B^2 - 4AC = 4 - 4xt$ so the operator is hyperbolic in $xt < 1$, parabolic on the hyperbola $xt = 1$ and elliptic in $xt > 1$.