Analytical Methods: Solutions 1

1. $\frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta f)+\frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}(v_\phi f)+\sin^2\theta\cos 2\phi=0$ with

$$v_\theta = a\sin\theta\cos\theta\cos 2\phi, \qquad v_\phi=-a\sin\theta\sin 2\phi.$$

First we need to substitute the v terms in and tidy up the equation:

$$\frac{\cos 2\phi}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\cos\theta f)-\frac{\partial}{\partial\phi}(\sin 2\phi f)+\sin^2\theta\cos 2\phi=0$$

$$\sin\theta\cos\theta\cos 2\phi\frac{\partial f}{\partial\theta}-\sin 2\phi\frac{\partial f}{\partial\phi}-3f\sin^2\theta\cos 2\phi+\sin^2\theta\cos 2\phi=0$$

Now we look for characteristics: curves on which

$$\sin\theta\cos\theta\cos 2\phi\frac{\partial}{\partial\theta}-\sin 2\phi\frac{\partial}{\partial\phi}=g(\theta,\phi)\frac{\mathrm{d}}{\mathrm{d}r}$$

We can decouple the equations by dividing through by $\cos 2\phi$ to give the two parametric equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=\sin\theta\cos\theta \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r}=-\frac{\sin 2\phi}{\cos 2\phi}$$

The ϕ equation integrates easily:

$$\int\frac{2\cos 2\phi}{\sin 2\phi}\,\mathrm{d}\phi=-2\int\mathrm{d}r \qquad \ln\sin 2\phi=-2r+C' \qquad \sin 2\phi=Ce^{-2r}.$$

The θ equation is a little harder:

$$\int\mathrm{d}r=\int\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\,\mathrm{d}\theta=\int\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\,\mathrm{d}\theta=\ln\sin\theta-\ln\cos\theta$$

Our characteristic is given parametrically by

$$\sin 2\phi=Ce^{-2r}, \qquad r=\ln\tan\theta, \qquad \sin 2\phi\tan^2\theta=C.$$

This curve satisfies the two equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=\sin\theta\cos\theta \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r}=-\frac{\sin 2\phi}{\cos 2\phi}$$

and so our original PDE becomes

$$\cos 2\phi\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\partial f}{\partial\theta}+\cos 2\phi\frac{\mathrm{d}\phi}{\mathrm{d}r}\frac{\partial f}{\partial\phi}-3f\sin^2\theta\cos 2\phi+\sin^2\theta\cos 2\phi=0$$

$$\frac{\mathrm{d}f}{\mathrm{d}r}-3f\sin^2\theta+\sin^2\theta=0$$

We need to substitute $\sin^2\theta$ in terms of r before solving:

$$\tan\theta=e^r \qquad \tan^2\theta=e^{2r} \qquad \cos^2\theta=\frac{1}{(1+e^{2r})} \qquad \sin^2\theta=\frac{e^{2r}}{(1+e^{2r})}$$

$$(1+e^{2r})\frac{\mathrm{d}f}{\mathrm{d}r} - 3e^{2r}f + e^{2r} = 0$$

This ODE has general solution

$$f = F(C)(1+e^{2r})^{3/2} + \frac{1}{3}$$

Finally we need to return to the original variables θ and ϕ, eliminating C and r from the solution. We already know $r = \ln\tan\theta$ and $C = \sin 2\phi \tan^2\theta$ so the final solution is

$$f = F(\sin 2\phi \tan^2\theta)\sec^3\theta + \frac{1}{3}.$$

2. $\dfrac{\partial u}{\partial t} + u^2\dfrac{\partial u}{\partial x} = 0.$

This is a nonlinear first-order PDE. We look for characteristics of the form

$$x = x(r) \qquad t = t(r) \qquad \text{along which } \frac{\mathrm{d}u}{\mathrm{d}r} = 0.$$

We look at the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u^2$$

for constant u, and see the curve family

$$x = u^2 r + x_0 \qquad t = r.$$

On each of these u is a constant, so u depends only on x_0 and not on r:

$$u = F(x_0).$$

We can rearrange the characteristic curve as $x_0 = x - u^2 t$ and thus the general implicit solution is

$$u = F(x - u^2 t).$$

Now we want to apply the initial conditions: $u(x,0) = \sqrt{x}$ gives

$$\sqrt{x} = F(x) \qquad u = \sqrt{(x - u^2 t)}.$$

The boundary condition $u(0,t) = 0$ is now automatically satisfied.

We can rearrange our implicit solution to make it explicit:

$$u = \sqrt{(x-u^2t)} \qquad u^2 = x - u^2 t \qquad u^2(1+t) = x \qquad u(x,t) = \sqrt{\frac{x}{(1+t)}}.$$

3. $y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1$, with $y(0) = 0$ and $y(\pi/2) = 0$.

Put $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2$:

$$\begin{array}{ccccccccc}
y_0'' & & & + & y_0 & & & = & 1 \\
\varepsilon y_1'' & + & 2\varepsilon y_0' & + & \varepsilon y_1 & & & = & 0 \\
\varepsilon^2 y_2'' & + & 2\varepsilon^2 y_1' & + & \varepsilon^2 y_2 & + & \varepsilon^2 y_0 & = & 0
\end{array}$$

Leading order: $y_0'' + y = 1$ gives $y_0 = 1 + A_0 \cos x + B_1 \sin x$.
Boundary conditions: $A_0 = -1$, $B_1 = -1$. The leading-order solution is

$$y_0 = 1 - \cos x - \sin x.$$

Order ε: $y_1'' + 2y_0' + y_1 = 0$ becomes $y_1'' + y_1 = -2\sin x + 2\cos x$.
The general solution is $y_1 = x\sin x + x\cos x + A_1 \cos x + B_1 \sin x$ and applying the boundary conditions gives $A_1 = 0$ and $B_1 = -\pi/2$:

$$y_1 = (x - \pi/2)\sin x + x\cos x.$$

Order ε^2: $y_2'' + 2y_1' + y_2 + y_0 = 0$ becomes

$$y_2'' + y_2 = -\sin x - (1-\pi)\cos x - 2x\cos x + 2x\sin x - 1.$$

After a little more work we obtain the general solution

$$y_2 = (\pi/2)x\sin x - 1 - (1/2)x^2 \sin x - (1/2)x^2 \cos x + A_2 \cos x + B_2 \sin x.$$

Applying the boundary conditions fixes $A_2 = 1$ and $B_2 = 1 - \pi^2/8$, so

$$y_2 = (\pi/2)x\sin x - 1 - (1/2)x^2 \sin x - (1/2)x^2 \cos x + \cos x + (1 - \pi^2/8)\sin x.$$

The first three terms of the solution are

$$\begin{aligned} y = 1 - \cos x - \sin x + \varepsilon[(x - \pi/2)\sin x + x\cos x] \\ - \varepsilon^2[1 + (x^2/2 - \pi x/2 - 1 + \pi^2/8)\sin x + (x^2/2 - 1)\cos x]. \end{aligned}$$

4. $I = \displaystyle\int_0^\varepsilon \frac{\mathrm{d}x}{(\varepsilon^2 - x^2 + \cos\varepsilon - \cos x)^{1/2}}$.
Make a change of variables $x = \varepsilon z$ to give

$$I = \int_0^1 \frac{\varepsilon\,\mathrm{d}z}{(\varepsilon^2 - \varepsilon^2 z^2 + \cos\varepsilon - \cos(\varepsilon z))^{1/2}}$$

and now expand the cosine terms, keeping terms up to order ε^4 (the "1" terms cancel):

$$\begin{aligned} I &= \int_0^1 \frac{\varepsilon\,\mathrm{d}z}{(\varepsilon^2 - \varepsilon^2 z^2 - \frac{1}{2}\varepsilon^2 + \varepsilon^4/24 - [-\frac{1}{2}\varepsilon^2 z^2 + \varepsilon^4 z^4/24] + O(\varepsilon^6))^{1/2}} \\ &= \int_0^1 \frac{\mathrm{d}z}{(1 - z^2 - 1/2 + \varepsilon^2/24 + z^2/2 - \varepsilon^2 z^4/24 + O(\varepsilon^4))^{1/2}} \\ &= \int_0^1 \frac{\sqrt{2}\,\mathrm{d}z}{(1 - z^2 + \varepsilon^2(1 - z^4)/12 + O(\varepsilon^4))^{1/2}} \\ &= \int_0^1 \frac{\sqrt{2}\,\mathrm{d}z}{(1 - z^2)^{1/2}(1 + \varepsilon^2(1 + z^2)/12 + O(\varepsilon^4))^{1/2}} \end{aligned}$$

Now we can expand the bracket $(1 + \varepsilon^2(1 + z^2)/12 + O(\varepsilon^4))^{-1/2}$:

$$\begin{aligned} I &= \int_0^1 \frac{\sqrt{2}\,\mathrm{d}z}{(1 - z^2)^{1/2}}(1 + \varepsilon^2(1 + z^2)/12 + O(\varepsilon^4))^{-1/2} \\ &= \int_0^1 \frac{\sqrt{2}\,\mathrm{d}z}{(1 - z^2)^{1/2}}(1 - \varepsilon^2(1 + z^2)/24 + O(\varepsilon^4)) \end{aligned}$$

From here to the end is just calculus: substitute $z = \sin\theta$ and after some manipulation we obtain:

$$I = \frac{\pi}{\sqrt{2}}\left(1 - \frac{\varepsilon^2}{16} + O(\varepsilon^4)\right).$$

5. $u_t + uu_x = 0$.

We scale $s = \varepsilon^a t$, $z = \varepsilon^b x$ and $v = \varepsilon^c u$. The equation becomes

$$\varepsilon^{a-c} v_s + \varepsilon^{b-2c} v v_z = 0,$$

so we have a balance if $c = b - a$. Then since the quantities

$$t^{-b/a} x \qquad t^{-c/a} u = t^{(a-b)/a} u$$

are invariant, we can pose a solution (with $b = ma$)

$$u = t^{(b-a)/a} f(\xi) = t^{m-1} f(\xi) \qquad \xi = t^{-b/a} x = t^{-m} x.$$

The resultant ODE is

$$[f(\xi) - m\xi] f'(\xi) + (m-1) f(\xi) = 0.$$

If we are given the inital condition

$$u(x, 1) = \frac{x + (x^2 - 1)^{1/2}}{2}$$

then that fixes the function

$$f(\xi) = \frac{\xi + (\xi^2 - 1)^{1/2}}{2}$$

and we determine m from the ODE: $m = 1/2$. Then our solution is

$$u = t^{-1/2}\left(\frac{xt^{-1/2} + (x^2 t^{-1} - 1)^{1/2}}{2}\right) = \frac{x/t + [(x/t)^2 - t^{-1}]^{1/2}}{2}.$$

Using the method of characteristics, the characteristic curves are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad t = r + 1 \qquad x = ur + x_0.$$

with implicit solution

$$u = F(x + u - ut).$$

The initial condition

$$u(x, 1) = \frac{x + (x^2 - 1)^{1/2}}{2} \quad \text{gives} \quad F(x) = \frac{x + (x^2 - 1)^{1/2}}{2}$$

with implicit solution

$$2u = (x + u - ut) + ((x + u - ut)^2 - 1)^{1/2}$$

which rearranges to

$$-4tu^2 + 4xu - 1 = 0 \qquad u = \frac{x \pm \sqrt{x^2 - t}}{2t}.$$

Taking the positive root in order to match the initial condition, we obtain the same solution as before:

$$u = \frac{x + \sqrt{x^2 - t}}{2t} = \frac{x/t + [(x/t)^2 - t^{-1}]^{1/2}}{2}.$$