Analytical Methods: Exercises 4

1. [Blasius boundary layer] Consider the steady Navier-Stokes equations:

$$\underline{\nabla} \cdot \underline{u} = 0 \qquad \underline{u} \cdot \underline{\nabla u} = -\underline{\nabla} p + \nabla^2 \underline{u},$$

   here made dimensionless using a typical velocity U, the fluid density ρ and the viscous lengthscale $L = \eta/\rho U$.

   Investigate flow past a semi-infinite flat plate (no natural lengthscale):

$$\underline{u} \to \underline{e}_x \ \text{ at } \infty; \qquad \underline{u} = 0 \ \text{ on } x \geq 0,\, y = 0.$$

   A dilation transformation is appropriate; expect the vertical velocity to be a smaller scale than the horizontal. You may find it easiest to work with a streamfunction $u = \partial\psi/\partial y$. The ODE which results can only be solved numerically.

2. Find the asymptotic behaviour of

$$J_\nu(\nu z) = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} \exp\left[\nu z \sinh t - \nu t\right] \, \mathrm{d}t$$

   for fixed real z with $0 < z < 1$ as $\nu \to \infty$.

Answers

1. The leading-order scalings are $u = U(\xi)$, $v = x^{-1/2} V(\xi)$ and $p = P_0$ (a constant), in which $\xi = x^{-1/2} y$. A streamfunction gives

$$\psi = x^{1/2} f(\xi) \ \text{ with } \ U(\xi) = f'(\xi), \ \ V(\xi) = \frac{1}{2}[\xi f'(\xi) - f(\xi)].$$

   and the resulting ODE is $2f'''(\xi) + f(\xi) f''(\xi) = 0$, with $f(0) = f'(0) = 0$ and $f'(\xi) \to 1$ as $\xi \to \infty$.

2. $J_\nu(\nu z) \sim \left(\dfrac{1}{2\pi\nu(1 - z^2)^{1/2}} \right)^{1/2} \exp\left[\nu \left((1 - z^2)^{1/2} - \operatorname{arccosh}(1/z) \right) \right].$