Analytical Methods: Exercises 3

1. Solve the following PDE with the boundary conditions given:

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{(t+1)^2}\frac{\partial^2 u}{\partial x^2} = 0 \qquad u(x,0) = u(1,t) = u(2,t) = 0.$$

2. Consider the following equation and boundary conditions:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$

$$u(-1,y) = u(1,y) = 0 \qquad u(x,1) = 1 - x^2 \qquad \varepsilon\frac{\partial u}{\partial y}(x,0) + u(x,0) = 0.$$

   (a) Calculate the first two terms of a regular perturbation expansion, ignoring the boundary condition at $y = 0$. Satisfy the other three boundary conditions at leading order only.

   (b) What scaling can be applied to y to find another solution? Calculate two terms of this solution, using only the $y = 0$ boundary condition.

   (c) Taking ε as a normal parameter (i.e. forgetting that it is small), find the full solution to the problem by separating variables. You need not determine all the coefficients in the sum; but find the general solution satisfying the x-boundary conditions.

   (d) Comment on the structure of your general solution when ε is small.

3. Find the image of the unit disc $|z - 1| \leq 1$ under the mapping $w = 1/z$.

4. Find the image of $-\pi/2 < x < \pi/2$, $0 < y < 1$ under $w = \sin z$.

5. Find the image of $-\pi/4 < x < \pi/4$, $-1 < y < 1$ under $w = \sin z$.

6. Solve the problem $\nabla^2 u = 0$ for $1 < r < e^\alpha$, $0 < \alpha < \pi$ with boundary conditions

$$\partial u/\partial r(1,\theta) = 0 \qquad \partial u/\partial r(e^\alpha,\theta) = \sin\theta \qquad u(r,0) = 0 \qquad u(r,\pi) = 0$$

   (a) by separation of variables, and

   (b) using the transformation $w = \ln z$.

7. Look at the problem

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} = \cos x$$

   with boundary conditions $f(0) = 0$, $f(\pi) = 1$. Find the two distinguished stretches for this equation. Calculate the first three terms of the regular expansion, and apply the boundary condition at π to determine the constants.

   Now apply your stretch near $x = 0$. Find the first three terms of the inner solution, and apply the boundary condition at $x = 0$ to determine some of the constants in this expansion.

   Finally use an intermediate variable to match your two expressions and determine the remaining constants.

8. Calculate three terms of the outer solution of

$$(1+\varepsilon)x^2 y' = \varepsilon((1-\varepsilon)xy^2 - (1+\varepsilon)x + y^3 + 2\varepsilon y^2) \quad \text{in } 0 < x < 1$$

   with $y(1) = 1$. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Thence find two terms for this inner solution.

Answers [Note: questions 1 and 3–6 are from Weinberger; 8 is from Hinch]

1. $u(x,t) = \sum_n \alpha_n x^{1/2}(t+1)^{1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) \sin\left(\frac{n\pi \ln(t+1)}{\ln 2}\right).$

2. (a) $u \sim 1 - x^2 + \varepsilon(2y + f_1(x)) + \cdots$
   (b) The scaling is $y = a + \varepsilon Y$ giving $f \sim A_0(x)e^{-Y} + \varepsilon A_1(x)e^{-Y} + \cdots$.
   (c) $u = \sum_n a_n \cos\left[\frac{(2n+1)\pi x}{2}\right] (c_n \exp[m_1 y] + d_n \exp[m_2 y])$ with $m_1,\ m_2 = [-1 \pm \sqrt{1 + (2n+1)^2\pi^2\varepsilon^2}]/2\varepsilon$.

3. $\mathrm{Real}\,(w) \geq 1/2$.

4. Putting $w = \eta + i\xi$, the image is $(\eta/\cosh 1)^2 + (\xi/\sinh 1)^2 \leq 1$, $\xi \geq 0$.

5. Putting $w = \eta + i\xi$, the image is the curvilinear rectangle bounded by the hyperbola $\eta^2 - \xi^2 = 1/2$ and the ellipse $(\eta/\cosh 1)^2 + (\xi/\sinh 1)^2 = 1$.

6. $u(r,\theta) = (r + r^{-1})\sin\theta/(1 - e^{-2\alpha})$.

7. $\delta = 1$, $\delta = \varepsilon$. Outer: $f = 1 + \sin x - \varepsilon[1 + \cos x] - \varepsilon^2 \sin x + \cdots$
   Inner ($x = \varepsilon z$): $f = a_0 - a_0 e^{-z} + \varepsilon[a_1 - a_1 e^{-z} + z] + \varepsilon^2[a_2 - a_2 e^{-z}] + \cdots$
   After matching: $f(z) = 1 - e^{-z} + \varepsilon[2e^{-z} - 2 + z] + O(\varepsilon^3)$.

8. Outer $y \sim 1 + \varepsilon[1 - 1/x] + \varepsilon^2[1/2 - 2/x + 3/(2x^2)] + \cdots$.
   Inner (with $x = \varepsilon z$): $y \sim (1 + 2/z)^{-1/2} + \varepsilon[(1 + 1/z)(1 + 2/z)^{-3/2}] + \cdots$