Analytical Methods: Exercises 2

1. Find the distinguished scalings, and the first two terms in the expansion of each root, for the following equation:

$$\varepsilon x^3 + x^2 + (2-\varepsilon)x + 1 = 0.$$

2. Find the first two terms of all four roots of $\varepsilon x^4 - x^2 - x + 2 = 0$.

3. Work out the first two terms in an expansion of each solution to $xe^{-x} = \varepsilon$.

4. Verify that the function

$$u = \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x',t')\,\mathrm{d}x'\,\mathrm{d}t'$$

satisfies the inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = F(x,t)$.

5. [Weinberger p.40] Find the characteristics through $(0,1)$ for the equation

$$\frac{\partial^2 u}{\partial t^2} - e^{2x}\frac{\partial^2 u}{\partial x^2} = 0.$$

6. Find two terms of a regular perturbation expansion for $f(x,t)$ in:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \varepsilon \cos x f = x$$

with boundary conditions $f(x,0) = \partial f/\partial t(x,0) = 0$. This particular problem can be solved in the same way even if $\varepsilon = 1$: this is the method of *successive approximations*. [Ref: Weinberger p. 384.]

7. $\varepsilon\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\dfrac{\mathrm{d}f}{\mathrm{d}x} - f = 0.$

   (a) Find the scalings $f = \varepsilon^\alpha F$ and stretches $x = a + \varepsilon^\beta z$ at which two dominant terms balance, and sketch these scalings in the α–β plane.

   (b) Hence determine the critical α and β for all three terms to balance.

   (c) Give also the possible values of β if the boundary conditions fix $\alpha = 0$. Find the leading term in an expansion for f in each case.

8. Find the distinguished stretches, and the leading term of each solution:

$$\varepsilon^3\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} + f = 0.$$

9. [Weinberger] Find where the following operators are hyperbolic, parabolic, and elliptic:

$$\text{(a)}\quad \frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} \qquad \text{(b)}\quad t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.$$

Answers

1. Scalings $x \sim 1$ and $x \sim \varepsilon^{-1}$; roots $x = -1-2\varepsilon+O(\varepsilon^2)$, $x = -\varepsilon^{-1}+2+O(\varepsilon)$, $x = -1$ (exact solution, no further terms).

2. $x \sim 1 + \varepsilon/3$; $x \sim -2 - 16\varepsilon/3$; $x \sim \varepsilon^{-1/2} + 1/2$; $x \sim -\varepsilon^{-1/2} + 1/2$.

3. $xe^{-x} = \varepsilon$. There are two roots: $x \sim \varepsilon + \varepsilon^2$ and $x \sim \ln(1/\varepsilon) - \ln(\ln(1/\varepsilon))$.

5. $t = e^x$ and $t = 2 - e^x$.

6. $f(x,t) = \frac{1}{2}xt^2 + \varepsilon\left[\frac{1}{2}t^2(x\cos x - 2\sin x) - x\cos x + 4\sin x + \frac{1}{2}(x+t)\cos(x+t) + \frac{1}{2}(x-t)\cos(x-t) - 2\sin(x+t) - 2\sin(x-t)\right]$

7. (a) $\alpha+\beta = 1$, $\alpha < \beta$; $\beta = 1/2$, $\alpha > \beta$; $\alpha = \beta$, $\beta < 1/2$. (b) $\alpha = \beta = 1/2$.
   (c) $\beta = 0$: $f_0 = a + x$. $\beta = 1$: any of $F_0 =$ constant, $F_0 = 2(z+b)^{-1}$, $F_0 = -2k\tan[k(z+b)]$, $2k\tanh[k(z+b)]$, or $2k\coth[k(z+b)]$.

8. 1, ε, ε^2. $f = be^{-x}$; $f = be^{-(x-a)/\varepsilon} + c$; $f = be^{-(x-a)/\varepsilon^2} + c(x-a)/\varepsilon^2 + d$.

9. (a) H $t^2 > 4x$; P $t^2 = 4x$; E $t^2 < 4x$. (b) H $xt < 1$; P $xt = 1$; E $xt > 1$.