Analytical Methods: Exercises 2

1. Find the distinguished scalings, and the first two terms in the expansion of each root, for the following equation:

\[ \varepsilon x^3 + x^2 + (2 - \varepsilon)x + 1 = 0. \]

2. Find the first two terms of all four roots of \( \varepsilon x^4 - x^3 - x + 2 = 0. \)

3. Work out the first two terms in an expansion of each solution to \( xe^{-x} = \varepsilon. \)

4. Verify that the function

\[ u = \frac{1}{2c} \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x', t') \, dx' \, dt' \]

satisfies the inhomogeneous wave equation \( u_{tt} - c^2 u_{xx} = F(x, t). \)

5. [Weinberger p.40] Find the characteristics through \((0, 1)\) for the equation

\[ \frac{\partial^2 u}{\partial t^2} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} = 0. \]

6. Find two terms of a regular perturbation expansion for \( f(x, t) \) in:

\[ \frac{\partial^2 f}{\partial t^2} - \varepsilon \frac{\partial^2 f}{\partial x^2} - \varepsilon \cos xf = x \]

with boundary conditions \( f(x, 0) = \partial f/\partial t(x, 0) = 0. \) This particular problem can be solved in the same way even if \( \varepsilon = 1: \) this is the method of successive approximations. [Ref: Weinberger p. 384.]

7. \[ \varepsilon \frac{d^2 f}{dx^2} + f \frac{df}{dx} - f = 0. \]

(a) Find the scalings \( f = \varepsilon^\alpha F \) and stretches \( x = a + \varepsilon^\beta z \) at which two dominant terms balance, and sketch these scalings in the \( \alpha - \beta \) plane.

(b) Hence determine the critical \( \alpha \) and \( \beta \) for all three terms to balance.

(c) Give also the possible values of \( \beta \) if the boundary conditions fix \( \alpha = 0. \) Find the leading term in an expansion for \( f \) in each case.

8. Find the distinguished stretches, and the leading term of each solution:

\[ \varepsilon^3 \frac{d^3 f}{dx^3} + \varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + f = 0. \]

9. [Weinberger] Find where the following operators are hyperbolic, parabolic, and elliptic:

\[ (a) \frac{\partial^2 u}{\partial t^2} + t \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} \quad (b) \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}. \]
Answers

1. Scalings $x \sim 1$ and $x \sim \varepsilon^{-1}$; roots $x = -1 - 2\varepsilon + O(\varepsilon^2)$, $x = -\varepsilon^{-1} + 2 + O(\varepsilon)$, $x = -1$ (exact solution, no further terms).

2. $x \sim 1 + \varepsilon/3; \ x \sim -2 - 16\varepsilon/3; \ x \sim \varepsilon^{-1}/2 + 1/2; \ x \sim -\varepsilon^{-1}/2 + 1/2$.

3. $xe^{-x} = \varepsilon$. There are two roots: $x \sim \varepsilon + \varepsilon^2$ and $x \sim \ln (1/\varepsilon) - \ln (\ln (1/\varepsilon))$.

4. $t = e^x$ and $t = 2 - e^x$.

5. $f(x, t) = \frac{1}{2}xt^2 + \varepsilon \left[ \frac{1}{2}t^2(x \cos x - 2 \sin x) - x \cos x + 4 \sin x + \frac{1}{2}(x + t) \cos (x + t) + \frac{1}{2}(x - t) \cos (x - t) - 2 \sin (x + t) - 2 \sin (x - t) \right]$

7. (a) $\alpha + \beta = 1$, $\alpha < \beta$; $\beta = 1/2$, $\alpha > \beta$; $\alpha = \beta$, $\beta < 1/2$. (b) $\alpha = \beta = 1/2$. (c) $\beta = 0$: $f_0 = a + x$. $\beta = 1$: any of $F_0 = constant$, $F_0 = 2(z + b)^{-1}$, $F_0 = -2k \tan [k(z + b)]$, $2k \tanh [k(z + b)]$, or $2k \coth [k(z + b)]$.

8. $1, \varepsilon, \varepsilon^2$. $f = be^{-x}$; $f = be^{-(x-a)/\varepsilon} + c$; $f = be^{-(x-a)/\varepsilon^2} + c(x-a)/\varepsilon^2 + d$.

9. (a) $H t^2 > 4x; \ P t^2 = 4x; \ E t^2 < 4x$. (b) $H xt < 1; \ P xt = 1; \ E xt > 1$. 