Analytical Methods: Exercises 1

1. Find the general solution to the PDE for $f(\theta, \phi)$:

$$
\frac{1}{a \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta f) + \frac{1}{a \sin \theta} \frac{\partial}{\partial \phi} (v_\phi f) + \sin^2 \theta \cos 2\phi = 0
$$

in which

$$
v_\theta = a \sin \theta \cos \theta \cos 2\phi
$$

$$
v_\phi = -a \sin \theta \sin 2\phi
$$

2. Consider the problem

$$
\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0
$$

in $x \geq 0$, $t \geq 0$, with initial and boundary conditions

$$
u(x, 0) = \sqrt{x} \quad u(0, t) = 0.
$$

Find the general solution implicitly and hence the specific solution in this case.

3. Try a regular perturbation expansion in the following differential equation:

$$
y'' + 2\varepsilon y' + (1 + \varepsilon^2)y = 1, \quad y(0) = 0, \quad y(\pi/2) = 0.
$$

Calculate the first three terms, that is, up to order $\varepsilon^2$. Apply the boundary conditions at each order.

4. Calculate the first two nonzero terms of a regular expansion in $\varepsilon$ for the following integral:

$$
I = \int_0^\varepsilon \frac{dx}{(\varepsilon^2 - x^2 + \cos \varepsilon - \cos x)^{1/2}}.
$$

[Hint: you will need to keep terms of order $\varepsilon^4$ initially.]

5. Try a dilation transformation on the Burger’s equation: $u_t + uu_x = 0$.

Find the specific solution for initial conditions

$$
u(x, 1) = \frac{x + (x^2 - 1)^{1/2}}{2}
$$

and show it matches that obtained by the method of characteristics.
Answers

1. \( f(\theta, \phi) = F(\sin 2\phi \tan^2 \theta) \sec^3 \theta + \frac{1}{3} \).

2. Implicit solution \( u = F(x - u^2t) \), particular solution \( u(x, t) = \sqrt{\frac{x}{1 + t}}. \)

3. 
   \[
y = 1 - \cos x - \sin x + \varepsilon [(x - \pi/2) \sin x + x \cos x] \\
   \quad - \varepsilon^2 [1 + (x^2/2 - \pi x/2 - 1 + \pi^2/8) \sin x + (x^2/2 - 1) \cos x].
   \]

4. \( I = \frac{\pi}{\sqrt{2}} \left( 1 - \frac{\varepsilon^2}{16} + O(\varepsilon^4) \right). \)

5. \( u = t^{m-1} f(\xi) \) with \( \xi = t^{-m} x \) and \( (f(\xi) - m\xi)f'(\xi) + (m - 1)f(\xi) = 0. \)
   Specific solution \( u = (x/t + [(x/t)^2 - t^{-1}])^{1/2}/2. \)