Analytical Methods: Exercises 1

1. Find the general solution to the PDE for $f(\theta, \phi)$:

$$\frac{1}{a \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta v_\theta f) + \frac{1}{a \sin\theta} \frac{\partial}{\partial\phi}(v_\phi f) + \sin^2\theta \cos 2\phi = 0$$

   in which

$$\begin{aligned} v_\theta &= a \sin\theta \cos\theta \cos 2\phi \\ v_\phi &= -a \sin\theta \sin 2\phi \end{aligned}$$

2. Consider the problem

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0$$

   in $x \geq 0$, $t \geq 0$, with initial and boundary conditions

$$u(x, 0) = \sqrt{x} \qquad u(0, t) = 0.$$

   Find the general solution implicitly and hence the specific solution in this case.

3. Try a regular perturbation expansion in the following differential equation:

$$y'' + 2\varepsilon y' + (1 + \varepsilon^2) y = 1, \qquad y(0) = 0, \qquad y(\pi/2) = 0.$$

   Calculate the first three terms, that is, up to order ε^2. Apply the boundary conditions at each order.

4. Calculate the first two nonzero terms of a regular expansion in ε for the following integral:

$$I = \int_0^\varepsilon \frac{\mathrm{d}x}{(\varepsilon^2 - x^2 + \cos\varepsilon - \cos x)^{1/2}}.$$

   [Hint: you will need to keep terms of order ε^4 initially.]

5. Try a dilation transformation on the Burger's equation: $u_t + u u_x = 0$. Find the specific solution for initial conditions

$$u(x, 1) = \frac{x + (x^2 - 1)^{1/2}}{2}$$

   and show it matches that obtained by the method of characteristics.

Answers

1. $f(\theta, \phi) = F(\sin 2\phi \tan^2 \theta) \sec^3 \theta + \dfrac{1}{3}$.

2. Implicit solution $u = F(x - u^2 t)$, particular solution $u(x, t) = \sqrt{\dfrac{x}{(1 + t)}}$.

3.

$$
\begin{aligned}
y = 1 - \cos x - \sin x + \varepsilon[(x - \pi/2) \sin x + x \cos x] \\
- \varepsilon^2[1 + (x^2/2 - \pi x/2 - 1 + \pi^2/8) \sin x + (x^2/2 - 1) \cos x].
\end{aligned}
$$

4. $I = \dfrac{\pi}{\sqrt{2}} \left(1 - \dfrac{\varepsilon^2}{16} + O(\varepsilon^4)\right)$.

5. $u = t^{m-1} f(\xi)$ with $\xi = t^{-m} x$ and $(f(\xi) - m\xi) f'(\xi) + (m - 1) f(\xi) = 0$.
   Specific solution $u = (x/t + [(x/t)^2 - t^{-1}]^{1/2})/2$.