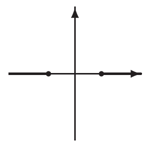
G Conformal maps

You will sometimes see these analytic functions referred to as conformal maps: in fact there is a subtle distinction. An analytic function provides a conformal map only if its derivative is nonzero throughout the domain. The meaning of *conformal map* is a map which preserves angles (though not necessarily lengths).

Most of the art of using conformal maps to improve problems involving Laplace's equation is in choosing the map to use. Here are a few domains and functions which improve them (starting with the example we just solved).

Two infinite plates

The geometry we just discussed took the plane with two branch cuts missing:



was transformed under the mapping

$$w(z) = z + \sqrt{(z^2 - 1)}$$

to the upper half plane.

Bilinear mapping

We can take the upper half plane $\mathrm{Imag}\,(z) \geq 0$ to the unit disc $|w| \leq 1$ using

$$w(z) = \frac{z - \alpha}{z - \overline{\alpha}}$$

where α is any constant with $\mathrm{Imag}\,(\alpha) > 0$.

Zhukovsky transform

In two-dimensional inviscid fluid mechanics with a steady velocity field (u, v), we can use a complex function

$$f(z) = \phi + i\psi \qquad z = x + iy$$

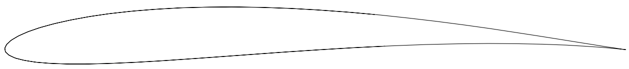
to represent the velocity:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = u + iv.$$

The Zhukovsky transform is one of the classics of early aerodynamics. Under the transformation:

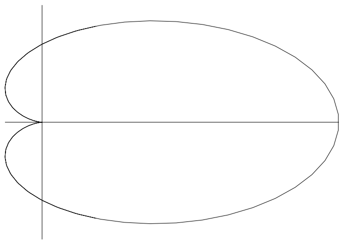
$$w(z) = z + \frac{1}{z}$$

a disc not centred on the origin, whose boundary passes through the point $z = 1$, transforms into the Zhukovsky aerofoil:



Cardioid

The cardioid $r = 2(1 + \cos\theta)$:



can be transformed into the unit disc $|w| < 1$ using the mapping

$$w(z) = \sqrt{z} - 1$$

with the branch cut of the square root drawn along the negative x-axis.

G.1 Boundary conditions

Suppose we map domain D_1 of z into a more pleasant domain D_2 of the w-plane, using the mapping $w(z)$. Then it is clear that if we have boundary conditions of the form

$$u = f(z) \qquad \text{on } \partial D_1$$

they can be transferred to w simply by using the inverse mapping:

$$U = f(z(w)) \qquad \text{on } \partial D_2.$$

However, if our original boundary conditions were formed in terms of the derivative normal to the boundary:

$$\frac{\partial u}{\partial n} = g(z) \qquad \text{on } \partial D_1$$

we need to work a little harder. For the details of how to work this out, see Weinberger p. 243; but in essence, the derivatives normal to the boundary in the two domains are directly related via the (complex) derivative of the mapping $w(z)$:

$$\frac{\partial u}{\partial n} = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| \frac{\partial U}{\partial n} \qquad \text{so} \qquad \frac{\partial U}{\partial n} = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-1} g(z(w)) \qquad \text{on } \partial D_2.$$