F Laplace's equation: Complex variables

Let's look at Laplace's equation in 2D, using Cartesian coordinates:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

It has no real characteristics because its discriminant is negative ($B^2 - 4AC = -4$). But if we ignore this technicality and allow ourselves a complex change of variables, we can benefit from the same structure of solution that worked for the wave equation. Introduce

$$\begin{aligned} \eta &= x + iy & \qquad x &= (\eta + \xi)/2 \\ \xi &= x - iy & \qquad y &= (\eta - \xi)/2i. \end{aligned}$$

Then the chain rule gives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

and the PDE becomes

$$4\frac{\partial}{\partial \eta}\frac{\partial f}{\partial \xi} = 0$$

whose solution is straightforward:

$$f = p(\eta) + q(\xi) = p(x + iy) + q(x - iy).$$

Here p and q are differentiable complex functions; and assuming we wanted a real solution to the original (real) PDE, we have an additional constraint that the sum of the two functions must have no imaginary part.

We can formalise this in more standard notation: if we use the (x, y) plane to represent the complex plane in the usual way, we introduce the complex variable $z = x + iy$. Then its complex conjugate is $\overline{z} = x - iy$ and the solution we have just found is

$$f = p(z) + q(\overline{z}).$$

F.1 Cauchy-Riemann Equations

Let's look at our function $p(\eta) = p(z)$, which forms half of our "characteristics"-style solution. It is obvious that

$$\frac{\partial p}{\partial \xi} = \frac{\partial p}{\partial \overline{z}} = 0$$

and using the chain rule, this tells us that

$$\frac{1}{2}\frac{\partial p}{\partial x} - \frac{1}{2i}\frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial x} = -i\frac{\partial p}{\partial y}.$$

Now if we divide the function into its real and imaginary parts:

$$p(z) = u(x, y) + iv(x, y)$$

where u and v are real functions, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

This complex equation is equivalent to the pair of real equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the **Cauchy-Riemann equations**, and are satisfied by the real and imaginary parts of any differentiable function of a complex variable $z = x + iy$.

In fact in a given domain, u and v (continuously differentiable) satisfy the Cauchy-Riemann equations if and only if p is an **analytic** function of z. We will not prove this here.

(Recall $f(z)$ is analytic $\equiv$ holomorphic within a domain D if, in every circle $|z - z_1| < \rho$ lying in D, f can be represented as a power series in $z - z_1$.)

F.2 General solution of Laplace's equation

We had the solution

$$f = p(z) + q(\overline{z})$$

in which $p(z)$ is analytic; but we can go further: remember that Laplace's equation in 2D can be written in polar coordinates as

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

and we showed by separating variables that in the whole plane (except the origin) it has solutions

$$f(r,\theta) = A + B\ln r + \sum_n (a_n \cos(n\theta) + b_n \sin(n\theta))(c_n r^n + d_n r^{-n}).$$

(In fact we also discarded some solutions which were not 2π-periodic in θ; these may be valid in a domain which does not encircle the origin.) Now in these variables, $z = r\exp[i\theta]$ so we can also write the solution we found as

$$f = \text{Real}\left(A + B\ln z + \sum_n \left[c_n(a_n - ib_n)z^n + d_n(a_n + ib_n)z^{-n}\right]\right)$$

meaning that our solution is the real part of a function of z only:

$$f = \text{Real}\,(g(z)).$$

Note that $g(z)$ as given here is analytic in any simply connected domain that does not include the origin; if $B = 0$ it is analytic everywhere except the origin, and if additionally $d_n = 0$, it is analytic everywhere.

We have shown that the real solution to Laplace's equation we had found is the real part of an analytic function of $z = x + iy$ in our domain; we can show

the converse very quickly from the Cauchy-Riemann equations. Consider an analytic function

$$f(z) = u(x, y) + iv(x, y)$$

Then the Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Differentiating the first w.r.t. x and the second w.r.t. y gives:

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We can solve Laplace's equation in any domain simply by taking the real part of any analytic function in that domain.

F.3 Composition of Analytic functions

The composition of two analytic functions is analytic (providing, of course, the relevant domains are correctly specified): if

$$f : D_1 \to D_2 \qquad \text{and} \qquad g : D_2 \to D_3$$

are both analytic, then the composed function

$$g \circ f : D_1 \to D_3$$

is also analytic on D_1.

This has important ramifications for the solution of Laplace's equation in odd-shaped domains or with boundary conditions which are unsuitable for separation of variables.

Suppose we are trying to find a real function u satisfying

$$\nabla^2 u = 0 \ \text{ in } \ D_1 \qquad \text{with } \ u = u(x, y) \ \text{ on } \ \partial D_1.$$

Of course this is equivalent to finding an analytic function $f(z)$ on D_1 whose real part satisfies the boundary condition on ∂D_1.

If D_1 is an awkward shape, and we can find an analytic function $w(z)$ which maps it to a more helpful domain D_2, then we can define

$$f = g \circ w \qquad f(z) = g(w(z))$$

and we are now looking for an analytic function g defined on D_2 such that

$$\text{Real}\,(g(w(z))) = u(z) \ \text{ on } \ \partial D_1.$$

$$\text{Real}\,(g(w)) = u(z(w)) \ \text{ on } \ \partial D_2.$$

Example

This is taken from an old UCL exam paper.

Find the solution to Laplace's equation in the domain D_1 given by the whole (x, y)-plane except for two semi-infinite plates $|x| \geq 1$, $y = 0$. The boundary conditions on these two plates are

$$u(x, 0) = 0 \quad \text{on} \quad x \geq 1; \qquad u(x, 0) = 1 \quad \text{on} \quad x \leq -1.$$

The domain looks superficially suitable for separation of variables in Cartesian coordinates, but the boundary conditions are not suitable: we would need $u(x, 0)$ to be prescribed for **all** x for separation to work.

Here we use the map $w(z) = z + \sqrt{(z^2 - 1)}$. Note that the square root means this map is not analytic over the whole plane; we need a branch cut at each of $z = 1$, $z = -1$. Given the domain we are trying to transform, it makes sense to put the branch cuts on $y = 0$ (or z real) and $|x| \geq 1$ (or $|z| \geq 1$).

The point $z = 0$ maps to $w = \sqrt{(-1)}$ and we can choose which of the possible values we take for the sign of the square root here: we choose $w(0) = i$. This choice, with the positioning of the branch cuts, determines $w(z)$ everywhere in our domain – in the diagram I've marked the result of each of the square roots at points around it. So when $z = x + i\varepsilon$ and $x > 1$, both roots are positive; when $z = x$ and $|x| < 1$, the root at $z = 1$ has argument i and the other is positive; when $z = x - i\varepsilon$ with $x < -1$, both roots have argument i so the product is negative, and so on:

$$\begin{array}{c} i \\ \hline -i \end{array} \bullet + \qquad i \bullet \begin{array}{c} + \\ \hline - \end{array}$$

In particular:

$w(-1) = -1$	$w(1) = 1$
$w(x) = x + i\sqrt{(1 - x^2)}$	$-1 < x < 1$
$w(x + i\varepsilon) = x - \sqrt{(x^2 - 1)}$	$x < -1$
$w(x - i\varepsilon) = x + \sqrt{(x^2 - 1)}$	$x < -1$
$w(x + i\varepsilon) = x + \sqrt{(x^2 - 1)}$	$x > 1$
$w(x - i\varepsilon) = x - \sqrt{(x^2 - 1)}$	$x > 1$

Thus the branch cuts in the z-plane map onto the real line in the w-plane: the left-hand cut maps to (top side) $w < -1$ and (bottom side) $-1 < w < 0$, and the right-hand cut maps to (top side) $w > 1$ and (bottom side) $0 < w < 1$. In the w-plane we now need to solve Laplace's equation for a new function v with

$$v(x, 0) = 1 \quad \text{on} \quad x \leq 0 \qquad v(x, 0) = 0 \quad \text{on} \quad x \geq 0.$$

This new problem is suitable for separation of variables in polar coordinates: the boundary conditions in terms of r and θ are

$$v(r, 0) = 0 \qquad v(r, \pi) = 1.$$

Note that our domain now does not encircle the origin, so we must revisit our separable solution and include some terms we discarded earlier.

We look for the form $v = R(r)T(\theta)$ and derive the coupled ODEs

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = A \qquad \frac{T''(\theta)}{T(\theta)} = -A.$$

In the three cases $A < 0$, $A > 0$ and $A = 0$ respectively these yield:

$$v = (A_\mu \exp[\mu\theta] + B_\mu \exp[-\mu\theta])(C_\mu \cos[\mu \ln r] + D_\mu \sin[\mu \ln r])$$
$$v = (a_\lambda \cos[\lambda\theta] + b_\lambda \sin[\lambda\theta])(c_\lambda r^\lambda + d_\lambda r^{-\lambda})$$
$$v = (\alpha + \beta \ln r)(\gamma + \delta\theta).$$

Applying the boundary condition $T(\theta = 0) = 0$ gives the three basis functions

$$v = \sinh[\mu\theta](C_\mu \cos[\mu \ln r] + D_\mu \sin[\mu \ln r])$$
$$v = \sin[\lambda\theta](c_\lambda r^\lambda + d_\lambda r^{-\lambda})$$
$$v = \theta(\alpha + \beta \ln r),$$

and the condition that v must be well-behaved at $r = 0$ (since the origin is in our domain) fixes further:

$$v = \alpha\theta + \sum_\lambda c_\lambda r^\lambda \sin[\lambda\theta]$$

The final boundary condition $v(r, \pi) = 1$ gives

$$1 = \alpha\pi + \sum_\lambda c_\lambda r^\lambda \sin[\lambda\pi]$$

which is satisfied with $\alpha = 1/\pi$ and $c_\lambda = 0$. Thus we have found

$$v(r, \theta) = \frac{\theta}{\pi}.$$

In order to convert this to a solution to our original problem, we first need to find the analytic function of which it is the real part. In this case the function is straightforward:

$$v(r, \theta) = \frac{\theta}{\pi} = -\frac{1}{\pi} \operatorname{Real}(i[\ln r + i\theta]) = -\frac{1}{\pi} \operatorname{Real}(i \ln w)$$

so the analytic function we need is

$$g(w) = -\frac{i \ln w}{\pi}.$$

Finally we need to convert back to the original variables:

$$f(z) = g \circ w(z) = -\frac{i}{\pi} \ln\left\{z + \sqrt{(z^2 - 1)}\right\}$$

and the solution we need is the real part of this:

$$u(x, y) = \operatorname{Real}\left(-\frac{i}{\pi} \ln\left\{z + \sqrt{(z^2 - 1)}\right\}\right) = \frac{1}{\pi} \operatorname{Imag}\left(\ln\left\{z + \sqrt{(z^2 - 1)}\right\}\right).$$

In particular, on the "missing line" $y = 0$, $-1 \le x \le 1$, we have

$$u(x, 0) = \frac{1}{\pi} \operatorname{Arg}\left(\left\{x + i\sqrt{(1 - x^2)}\right\}\right) = \frac{1}{\pi} \arctan \frac{\sqrt{(1 - x^2)}}{x}.$$