E Separation of variables: a “lucky” method

Let’s look now at the most general constant-coefficient homogeneous linear PDE of second order:

\[ A \frac{\partial^2 f}{\partial t^2} + B \frac{\partial^2 f}{\partial x \partial t} + C \frac{\partial^2 f}{\partial x^2} + D \frac{\partial f}{\partial t} + E \frac{\partial f}{\partial x} + F f = 0. \]

If we can eliminate the mixed-derivative term then we have a chance of using the method of separation of variables.

The linear change of variables we were looking at while classifying our equations:

\[ \xi = \alpha x + \beta t \quad \eta = \gamma x + \delta t \]

gave the mixed-derivative term as

\[ [2A\beta \delta + B(\alpha \delta + \beta \gamma) + 2C\alpha \gamma] \frac{\partial^2 f}{\partial \xi \partial \eta} \]

It is clear that our four variables are more than enough: we can make a choice under which there is no mixed-derivative term. We’ll look later at how to optimise the choice.

E.1 The basics

You will all have seen this method before: I will only run through it briefly. We seek to express our solution as a sum of solutions of the form

\[ f(x, t) = X(x)T(t). \]

Substituting this into the governing equation (we’ve made our change of variables already so there is no mixed derivatives term)

\[ A \frac{\partial^2 f}{\partial t^2} + C \frac{\partial^2 f}{\partial x^2} + D \frac{\partial f}{\partial t} + E \frac{\partial f}{\partial x} + F f = 0 \]

gives

\[ AX''(x)T(t) + CX''(x)T(t) + DX(x)T'(t) + EX'(x)T(t) + FX(x)T(t) = 0 \]

\[ \frac{AT''(t)}{T(t)} + \frac{DT'(t)}{T(t)} = -\frac{CX''(x)}{X(x)} - \frac{EX'(x)}{X(x)} - F \]

Now the left hand side of this equation is a function of \( t \) only and the right hand side only depends on \( x \), so they must both be a constant, \( \lambda \), independent of \( x \) and \( t \). This insight gives us two ODEs to solve:

\[ AT''(t) + DT'(t) - \lambda T(t) = 0 \quad CX''(x) + EX'(x) + (F + \lambda)X(x) = 0. \]

These give us pairs of solutions, coupled through the value of the constant \( \lambda \), and typically we write the final solution as

\[ f(x, t) = \sum_n X_n(\lambda_n, x)T_n(\lambda_n, t). \]
Example: Laplace in plane polars

Laplace’s equation in plane polar coordinates is

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = 0 \quad \text{and} \quad r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0.
\]

The separable solution \( f(r, \theta) = R(r)T(\theta) \) gives the coupled ODEs

\[
\frac{r^2 R''(r) + r R'(r)}{R(r)} = A \quad \text{and} \quad \frac{T''(\theta)}{T(\theta)} = -A.
\]

We look at the three cases \( A > 0 \), \( A < 0 \) and \( A = 0 \) separately; and because we’re in polar coordinates, any solution must be periodic of period 2\( \pi \) in \( \theta \).

**Positive constant \( A = \lambda^2 \)**

\[
r^2 R''(r) + r R'(r) - \lambda^2 R(r) = 0.
\]

\( T''(\theta) = -\lambda^2 T(\theta) \) gives \( T(\theta) = b_1 \cos \lambda \theta + b_2 \sin \lambda \theta \)

and the periodicity condition fixes \( \lambda \) to be an integer.

**Negative constant \( A = -\mu^2 \)**

\[
T''(\theta) = \mu^2 T(\theta)
\]

\( T(\theta) = c_1 e^{i\mu \theta} + c_2 e^{-i\mu \theta} \)

and now the periodicity condition cannot be satisfied for \( \mu \neq 0 \).

**Zero constant \( A = 0 \)**

\[
T''(\theta) = 0 \quad T(\theta) = d_1 + d_2 \theta \quad d_2 = 0.
\]

\[
r^2 R''(r) + r R'(r) = 0 \quad R(r) = d_3 + d_4 \ln r.
\]

The general solution to Laplace’s equation in plane polars is then:

\[
f(r, \theta) = A + B \ln r + \sum_n \left( a_n \cos n \theta + b_n \sin n \theta \right) \left( c_n r^n + d_n r^{-n} \right).
\]

E.2 Boundary conditions

Of course, Laplace’s equation is also separable (has no mixed derivatives) in Cartesian coordinates; and a similar procedure produces the general solution

\[
f(x, y) = (\alpha x + \beta)(\gamma y + \delta)
\]

\[
+ \int \left( a(\lambda) \cos \lambda x + b(\lambda) \sin \lambda x \right) \left( c(\lambda) e^{\lambda y} + d(\lambda) e^{-\lambda y} \right) \, d\lambda
\]

\[
+ \int \left( A(\lambda) \cos \lambda y + B(\lambda) \sin \lambda y \right) \left( C(\lambda) e^{\lambda x} + D(\lambda) e^{-\lambda x} \right) \, d\lambda
\]

so how do we know which solution to use?

The simple answer is that the boundary conditions are crucial. Any second order PDE possesses a range of possible coordinates in which it has no mixed derivatives; and the boundary conditions of the specific problem to be solved must inform our choice.
We need the following conditions to be satisfied:

Separable equation
The differential equation must be separable: that is, there are no mixed derivatives and, if the coefficients depend on \( \eta \) and \( \xi \), then (after multiplication of the whole equation by some function if necessary) the derivatives w.r.t. \( \eta \) have coefficients which depend only on \( \eta \) and those w.r.t. \( \xi \) have coefficients which depend only on \( \xi \). The coefficient of the no-derivatives term must be at worst the sum of a function of \( \eta \) and a function of \( \xi \).

\[
\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{(t+1)^2} \frac{\partial^2 u}{\partial x^2} = 0 \text{ is OK} \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \cos(xt)u = 0 \text{ is not.}
\]

Boundary conditions on coordinate lines
All the boundary conditions in the problem must be located along lines \( \eta = \text{constant} \) or \( \xi = \text{constant} \). This does include the possibility of a boundary condition as one variable \( \to \infty \).

Correct type of boundary conditions
Along a line \( \eta = \text{constant} \), the boundary condition must not involve any partial derivatives with respect to \( \xi \); and the coefficients of derivatives involved in the boundary conditions must not vary with \( \xi \).

\[
\frac{\partial f}{\partial \eta}(0, \xi) = g(\xi) \text{ is OK} \quad \left(\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \xi}\right)(0, \xi) = 0 \text{ is not.}
\]

The equivalent condition is required of the boundary conditions along a line \( \xi = \text{constant} \).

Realistically, the boundary conditions are likely to completely constrain the co-ordinates we use if we wish to use separation of variables; and if the coordinates that work for the boundary conditions don’t work for the PDE, there’s very little we can do about it.

Example
[Weinberger p. 70.]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0
\]

This is a flukey one: it looks like it won’t work but a bit of cunning will get us there. First we try the standard separable solution:

\[
u = X(x)Y(y) \quad X''(x)Y(y) + X'(x)Y'(y) + X(x)Y''(y) = 0
\]

and then look at \( Y''/Y \):

\[
\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{X'(x)Y'(y)}{X(x)Y(y)}.
\]
Taking the partial derivative of this equation w.r.t. $x$ (and noting that the left hand side is independent of $x$) gives

$$0 = \frac{d}{dx} \left( \frac{X''(x)}{X(x)} \right) + \frac{Y'(y)}{Y(y)} \frac{d}{dx} \left( \frac{X'(x)}{X(x)} \right)$$

$$= \frac{X'''(x)X(x) - X''(x)X'(x)}{X^2(x)} + \frac{Y'(y)}{Y(y)} \left( \frac{X''(x)X(x) - X'(x)^2}{X^2(x)} \right),$$

which is separable if we divide by the bracketed term on the right:

$$- \frac{X'''(x)X(x) - X''(x)X'(x)}{X''(x)X(x) - X'(x)^2} = \frac{Y'(y)}{Y(y)} = 2\lambda$$

Now we proceed as before: solve

$$Y'(y) = 2\lambda Y(y) \quad Y(y) = e^{2\lambda y}$$

If we were to carry on with this equation we would have to solve

$$X'''(x)X(x) - X''(x)X'(x) + 2\lambda X''(x)X(x) - 2\lambda X'(x)^2 = 0$$

but now that we know $Y$, we can return to the original equation:

$$- \frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{X'(x)Y'(y)}{X(x)Y(y)}; \quad -4\lambda^2 = \frac{X''(x)}{X(x)} + 2\lambda \frac{X'(x)}{X(x)}$$

$$X(x) = e^{-\lambda x} (a \cos \sqrt{3}\lambda x + b \sin \sqrt{3}\lambda x)$$

and the general solution is

$$u(x, y) = \sum_{\lambda} \exp \left[ \frac{\lambda(2y - x)}{2} \right] (a_{\lambda} \cos \sqrt{3}\lambda x + b_{\lambda} \sin \sqrt{3}\lambda x).$$

The moral of this story is: if your boundary conditions look suitable for separation of variables, but your equation doesn’t, don’t despair – at least not until you’ve had a go!