E Separation of variables: a "lucky" method

Let's look now at the most general constant-coefficient homogeneous linear PDE of second order:

$$A\frac{\partial^2 f}{\partial t^2} + B\frac{\partial^2 f}{\partial x \partial t} + C\frac{\partial^2 f}{\partial x^2} + D\frac{\partial f}{\partial t} + E\frac{\partial f}{\partial x} + Ff = 0.$$

If we can eliminate the mixed-derivative term then we have a chance of using the method of **separation of variables**.

The linear change of variables we were looking at while classifying our equations:

$$\xi = \alpha x + \beta t \qquad \eta = \gamma x + \delta t$$

gave the mixed-derivative term as

$$[2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma]\frac{\partial^2 f}{\partial \xi \partial \eta}$$

It is clear that our four variables are more than enough: we can make a choice under which there is no mixed-derivative term. We'll look later at how to optimise the choice.

E.1 The basics

You will all have seen this method before: I will only run through it briefly. We seek to express our solution as a sum of solutions of the form

$$f(x,t) = X(x)T(t).$$

Substituting this into the governing equation (we've made our change of variables already so there is no mixed derivatives term)

$$A\frac{\partial^2 f}{\partial t^2} + C\frac{\partial^2 f}{\partial x^2} + D\frac{\partial f}{\partial t} + E\frac{\partial f}{\partial x} + Ff = 0$$

gives

$$AX(x)T''(t) + CX''(x)T(t) + DX(x)T'(t) + EX'(x)T(t) + FX(x)T(t) = 0$$

$$\frac{AT''(t)}{T(t)} + \frac{DT'(t)}{T(t)} = -\frac{CX''(x)}{X(x)} - \frac{EX'(x)}{X(x)} - F$$

Now the left hand side of this equation is a function of t only and the right hand side only depends on x, so they must both be a constant, λ, independent of x and t. This insight gives us two ODEs to solve:

$$AT''(t) + DT'(t) - \lambda T(t) = 0 \qquad CX''(x) + EX'(x) + (F + \lambda)X(x) = 0.$$

These give us pairs of solutions, coupled through the value of the constant λ, and typically we write the final solution as

$$f(x,t) = \sum_n X_n(\lambda_n, x)T_n(\lambda_n, t).$$

Example: Laplace in plane polars

Laplace's equation in plane polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}=0 \qquad r^2\frac{\partial^2 f}{\partial r^2}+r\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial \theta^2}=0.$$

The separable solution $f(r,\theta)=R(r)T(\theta)$ gives the coupled ODEs

$$\frac{r^2R''(r)+rR'(r)}{R(r)}=A \qquad \frac{T''(\theta)}{T(\theta)}=-A.$$

We look at the three cases $A>0$, $A<0$ and $A=0$ separately; and because we're in polar coordinates, any solution must be periodic of period 2π in θ.

Positive constant $A=\lambda^2$

$$r^2R''(r)+rR'(r)-\lambda^2R(r)=0. \qquad \text{gives} \qquad R(r)=a_1r^{\lambda}+a_2r^{-\lambda}.$$

$$T''(\theta)=-\lambda^2T(\theta) \qquad \text{gives} \qquad T(\theta)=b_1\cos\lambda\theta+b_2\sin\lambda\theta$$

and the periodicity condition fixes λ to be an integer.

Negative constant $A=-\mu^2$

$$T''(\theta)=\mu^2T(\theta) \qquad \text{gives} \qquad T(\theta)=c_1e^{\mu\theta}+c_2e^{-\mu\theta}$$

and now the periodicity condition cannot be satisfied for $\mu\neq 0$.

Zero constant $A=0$

$$T''(\theta)=0 \qquad T(\theta)=d_1+d_2\theta \qquad d_2=0.$$

$$r^2R''(r)+rR'(r)=0 \qquad R(r)=d_3+d_4\ln r.$$

The general solution to Laplace's equation in plane polars is then:

$$f(r,\theta)=A+B\ln r+\sum_n(a_n\cos n\theta+b_n\sin n\theta)(c_nr^n+d_nr^{-n}).$$

E.2 Boundary conditions

Of course, Laplace's equation is also separable (has no mixed derivatives) in Cartesian coordinates; and a similar procedure produces the general solution

$$\begin{aligned}f(x,y)&=(\alpha x+\beta)(\gamma y+\delta)\\&\quad+\int(a(\lambda)\cos\lambda x+b(\lambda)\sin\lambda x)(c(\lambda)e^{\lambda y}+d(\lambda)e^{-\lambda y})\,\mathrm{d}\lambda\\&\quad+\int(A(\lambda)\cos\lambda y+B(\lambda)\sin\lambda y)(C(\lambda)e^{\lambda x}+D(\lambda)e^{-\lambda x})\,\mathrm{d}\lambda\end{aligned}$$

so how do we know which solution to use?

The simple answer is that the boundary conditions are crucial. Any second order PDE possesses a range of possible coordinates in which it has no mixed derivatives: and the boundary conditions of the specific problem to be solved must inform our choice.

We need the following conditions to be satisfied:

Separable equation

The differential equation must be separable: that is, there are no mixed derivatives and, if the coefficients depend on η and ξ, then (after multiplication of the whole equation by some function if necessary) the derivatives w.r.t. η have coefficients which depend only on η and those w.r.t. ξ have coefficients which depend only on ξ. The coefficient of the no-derivatives term must be at worst the sum of a function of η and a function of ξ.

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{(t+1)^2}\frac{\partial^2 u}{\partial x^2} = 0 \;\text{ is OK} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \cos(xt)u = 0 \;\text{ is not.}$$

Boundary conditions on coordinate lines

All the boundary conditions in the problem must be located along lines η = constant or ξ = constant. This does include the possibility of a boundary condition as one variable $\to \infty$.

Correct type of boundary conditions

Along a line η = constant, the boundary condition must not involve any partial derivatives with respect to ξ; and the coefficients of derivatives involved in the boundary conditions must not vary with ξ.

$$\frac{\partial f}{\partial \eta}(0,\xi) = g(\xi) \;\text{ is OK} \qquad \left(\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \xi}\right)(0,\xi) = 0 \;\text{ is not.}$$

The equivalent condition is required of the boundary conditions along a line ξ = constant.

Realistically, the boundary conditions are likely to completely constrain the coordinates we use if we wish to use separation of variables; and if the coordinates that work for the boundary conditions don't work for the PDE, there's very little we can do about it.

Example

[Weinberger p. 70.]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is a flukey one: it looks like it won't work but a bit of cunning will get us there. First we try the standard separable solution:

$$u = X(x)Y(y) \qquad X''(x)Y(y) + X'(x)Y'(y) + X(x)Y''(y) = 0$$

and then look at Y''/Y:

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{X'(x)Y'(y)}{X(x)Y(y)}.$$

Taking the partial derivative of this equation w.r.t. x (and noting that the left hand side is independent of x) gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{X''(x)}{X(x)}\right) + \frac{Y'(y)}{Y(y)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{X'(x)}{X(x)}\right) \\ &= \frac{X'''(x)X(x) - X''(x)X'(x)}{X^2(x)} + \frac{Y'(y)}{Y(y)}\left(\frac{X''(x)X(x) - X'(x)^2}{X^2(x)}\right), \end{aligned}$$

which is separable if we divide by the bracketed term on the right:

$$-\frac{X'''(x)X(x) - X''(x)X'(x)}{X''(x)X(x) - X'(x)^2} = \frac{Y'(y)}{Y(y)} = 2\lambda$$

Now we proceed as before: solve

$$Y'(y) = 2\lambda Y(y) \qquad Y(y) = e^{2\lambda y}$$

If we were to carry on with this equation we would have to solve

$$X'''(x)X(x) - X''(x)X'(x) + 2\lambda X''(x)X(x) - 2\lambda X'(x)^2 = 0$$

but now that we know Y, we can return to the original equation:

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{X'(x)Y'(y)}{X(x)Y(y)} : \qquad -4\lambda^2 = \frac{X''(x)}{X(x)} + 2\lambda\frac{X'(x)}{X(x)}.$$

$$X(x) = e^{-\lambda x}(a\cos\sqrt{3}\lambda x + b\sin\sqrt{3}\lambda x)$$

and the general solution is

$$u(x,y) = \sum_\lambda \exp\left[\lambda(2y - x)\right](a_\lambda\cos\sqrt{3}\lambda x + b_\lambda\sin\sqrt{3}\lambda x).$$

The moral of this story is: if your boundary conditions look suitable for separation of variables, but your equation doesn't, don't despair – at least not until you've had a go!