C The second-order 1D wave equation

C.1 Homogeneous wave equation with constant speed

The simplest form of the second-order wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Like the first-order wave equation, it responds well to a change of variables:

$$\xi = x + ct \qquad \eta = x - ct$$

which reduces it to

$$-4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

which is solved by

$$u = p(\xi) + q(\eta) = p(x + ct) + q(x - ct)$$

for any differentiable functions p and q. The lines $\xi =$ constant and $\eta =$ constant are the characteristics, exactly analogous to the characteristics for the first-order equation.

If we add initial conditions

$$u(x, 0) = f(x) \qquad \partial u/\partial t(x, 0) = g(x)$$

then a little algebra gives us **d'Alembert's solution**:

$$u(x, t) = \frac{1}{2}\left[f(x + ct) + f(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y.$$

C.2 Inhomogeneous wave equation

The inhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x, t) \tag{3}$$

can be solved in a very similar way. The change of variables results in:

$$-4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = F\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right)$$

which can be integrated directly for any specific function F; however (Weinberger p. 25) it is also possible to carry out the integrals symbolically (paying particular attention to which variable is held constant when integrating with respect to another). The general result is

$$u = p(x + ct) + q(x - ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x', t')\,\mathrm{d}x'\,\mathrm{d}t'. \tag{4}$$

As this is a linear PDE, the general solution to the inhomogeneous equation is the sum of the general solution to the homogeneous equation (the *CF* in the notation of ODEs) and one particular solution to the full equation (the *PI*). To verify the solution we simply check that the last term of (4) satisfies (3).

Example

Let's consider the example inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 12xt.$$

We can find the solution using the formula (4) but it's not straightforward!

$$u = p(x+ct) + q(x-ct) + \frac{1}{2c} \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} 12x't' \, \mathrm{d}x' \, \mathrm{d}t'.$$

Looking just at the integral and carrying out the x' integration first gives

$$\begin{aligned} \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} 12x't' \, \mathrm{d}x' \, \mathrm{d}t' &= \int_0^t \left[6(x')^2 t'\right]_{x'=x-c(t-t')}^{x+c(t-t')} \, \mathrm{d}t' \\ &= \int_0^t 6\left((x+c(t-t'))^2 t' - (x-c(t-t'))^2 t'\right) \, \mathrm{d}t' \\ &= \int_0^t 24cxt'(t-t') \, \mathrm{d}t' = \left[cx(12tt'^2 - 8t'^3)\right]_{t'=0}^{t} \\ &= 4cxt^3, \end{aligned}$$

and the general solution is

$$u = p(x+ct) + q(x-ct) + 2xt^3.$$

For a specific case, though, it is usually more straightforward to work directly from the original equation (with change of variables). In this case when we put $\xi = x + ct$ and $\eta = x - ct$ we obtain

$$-4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{3}{c}(\xi+\eta)(\xi-\eta) = \frac{3}{c}(\xi^2 - \eta^2).$$

Integrating gives

$$\begin{aligned} -4c^2 \frac{\partial u}{\partial \xi} &= p(\xi) + \frac{1}{c}(3\xi^2\eta - \eta^3); \\ -4c^2 u &= p(\xi) + q(\eta) + \frac{\xi\eta}{c}(\xi+\eta)(\xi-\eta). \end{aligned}$$

Converting the coordinates gives

$$\begin{aligned} u &= f(x+ct) + g(x-ct) - \frac{(x+ct)(x-ct)}{4c^3}(2x)(2ct) \\ &= f(x+ct) + g(x-ct) - \frac{xt(x^2 - c^2t^2)}{c^2}. \end{aligned}$$

This doesn't immediately look the same; but note that the difference can be absorbed into $f(x+ct)$ and $g(x-ct)$:

$$\begin{aligned} 2xt^3 + \frac{xt(x^2 - c^2t^2)}{c^2} &= \frac{\left(xt(x^2 + c^2t^2)\right)}{c^2} = \frac{(\xi+\eta)(\xi-\eta)((\xi+\eta)^2 + (\xi-\eta)^2)}{64c^3} \\ &= \frac{1}{32c^3}\left((\xi^2 - \eta^2)(\xi^2 + \eta^2)\right) = \frac{1}{32c^3}(\xi^4 - \eta^4) \end{aligned}$$

which is the sum of a function of ξ and a function of η.

If you're in any doubt about your solution, plug it back into the original equation: as long as you have the p and q terms, anything that works will be the general solution!

C.3 Varying speed: two sets of characteristics

We saw in the constant-speed case that the characteristic curves were the straight lines

$$x = k_1 + ct \qquad x = k_2 - ct$$

Thus any point (x, t) lies on **two** characteristics, and finding the curves is not quite as straightforward as it was with the first-order wave equation. Characteristics, even for the homogeneous wave equation, are no longer curves along which u is constant.

To understand the wave equation better, let's look at the generalisation to a wavespeed which varies in space:

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0.$$

The characteristics in this case are curves which satisfy

$$\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 = \frac{1}{c^2(x)}.$$

Not everything carries over from the constant-speed case: the "obvious" change of variables

$$\xi = \int^x \frac{\mathrm{d}x'}{c(x')} + t \qquad \eta = \int^x \frac{\mathrm{d}x'}{c(x')} - t,$$

which makes the characteristics into lines of constant ξ or constant η, only reduces the governing equation to

$$-4\frac{\partial^2 u}{\partial\xi\partial\eta} - c'(x)\left(\frac{\partial u}{\partial\eta} + \frac{\partial u}{\partial\xi}\right) = 0,$$

which has no straightforward solution.

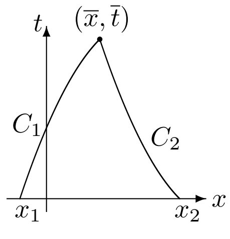
Suppose we specify our initial conditions:

$$u(x, 0) = f(x) \qquad \partial u/\partial t(x, 0) = g(x).$$

This time we will look at the value of the solution at a specific position and time $u(\overline{x}, \overline{t})$. We will prove that the solution depends only on the initial conditions over a range of x determined by the characteristics through our point: so that information propagates along the characteristic curves as in our previous cases.

The characteristics of this problem are curves which satisfy:

$$\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 = \frac{1}{c^2(x)} \qquad \frac{\mathrm{d}t}{\mathrm{d}x} = \pm\frac{1}{c(x)}.$$



We can find the two characteristic curves C_1 and C_2 passing through our point $(\overline{x}, \overline{t})$. These characteristics will reach the initial line $t = 0$ at points x_1 and x_2 respectively (we take $x_1 < x_2$ so that C_1 has positive $\mathrm{d}t/\mathrm{d}x$ and C_2 the negative sign). Either x_1 or x_2 may be infinite.

Now consider two different sets of initial conditions:

$$u_1(x,0) = f_1(x) \qquad u_2(x,0) = f_2(x)$$

$$\partial u_1/\partial t(x,0) = g_1(x) \qquad \partial u_2/\partial t(x,0) = g_2(x)$$

with $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$ over the range $x_1 \le x \le x_2$. If we can show that the corresponding solutions are equal at $(\overline{x}, \overline{t})$ then we know that the function value only depended on the initial conditions between $x_1 < x < x_2$.

The linearity of the problem means that the function $v(x,t) = u_1 - u_2$ satisfies

$$\frac{\partial^2 v}{\partial t^2} - c^2(x)\frac{\partial^2 v}{\partial x^2} = 0 \tag{5}$$

$$v(x,0) = f_1(x) - f_2(x) \qquad \partial v/\partial t(x,0) = g_1(x) - g_2(x)$$

with the initial condition functions both being zero over $x_1 \le x \le x_2$.

Multiplying (5) by $(1/c^2(x))(\partial v/\partial t)$ we can rewrite it as

$$\frac{\partial}{\partial t}\left[\frac{1}{2c^2(x)}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\right] = 0$$

Now we integrate this equation over the region between the line $t = 0$ and the two characteristics C_1 and C_2, meeting at the point $(\overline{x}, \overline{t})$: we integrate the first term first with respect to t and then x, and the second in the other order.

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \frac{1}{2c^2(x)}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x - \underbrace{\int_0^{\overline{t}}}_{C_2} \frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\,\mathrm{d}t + \underbrace{\int_0^{\overline{t}}}_{C_1} \frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\,\mathrm{d}t = 0$$

The later two can be converted to integrals over x, as we know $\mathrm{d}t/\mathrm{d}x$ on the two characteristics:

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \frac{1}{2c^2(x)}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = 0$$

and completing the square gives

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \left\{\frac{1}{2c^2(x)}\left[\frac{\partial v}{\partial t} + c^2(x)\frac{\partial v}{\partial x}\frac{\mathrm{d}t}{\mathrm{d}x}\right]^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\left[1 - c^2(x)\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right]\right\}\mathrm{d}x = 0$$

Finally, since $\mathrm{d}t/\mathrm{d}x = 1/c^2(x)$ on the characteristics, the second term is zero and we have

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \left\{\frac{1}{2c^2(x)}\left[\frac{\partial v}{\partial t} + c^2(x)\frac{\partial v}{\partial x}\frac{\mathrm{d}t}{\mathrm{d}x}\right]^2\right\}\mathrm{d}x = 0$$

Since the integrand is nonnegative, it must be zero along both characteristics. It follows that

$$\frac{\partial v}{\partial t} + c(x)\frac{\partial v}{\partial x} = 0 \text{ on } C_1; \qquad \frac{\partial v}{\partial t} - c(x)\frac{\partial v}{\partial x} = 0 \text{ on } C_2.$$

Since $(\overline{x}, \overline{t})$ lies on both characteristics it follows that $\partial v/\partial t = 0$ at $(\overline{x}, \overline{t})$.

Following the same procedure for a point $(\overline{x}, t_0)$ with $t_0 < \overline{t}$, the characteristics will lie within the triangle we used, and will intersect the line $t = 0$ inside the region $x_1 < x < x_2$ where the initial conditions are zero. Thus the working follows identically and we can deduce

$$\frac{\partial v}{\partial t}(\overline{x}, t) = 0 \text{ for } 0 \leq t \leq \overline{t}.$$

Since $v(\overline{x}, 0) = 0$ we can integrate wrt t to show $v(\overline{x}, \overline{t}) = 0$.

This completes the proof that $u_1(\overline{x}, \overline{t}) = u_2(\overline{x}, \overline{t})$ and the solution of the wave equation at $(\overline{x}, \overline{t})$ is only affected by information from those initial conditions lying within the characteristics through that point.