## 9 Integrals and Steepest Descents

### 9.1 Small parameter in the integration limits

In the last example, we needed the asymptotic behaviour of

$$
I(\rho)=\int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau^{2}} \mathrm{~d} \tau
$$

for small $\rho$. How did we find it? The procedure is actually quite straightforward: two steps of integration by parts give

$$
I(\rho)=\frac{e^{-\rho}}{\rho}-\int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau} \mathrm{d} \tau=\frac{e^{-\rho}}{\rho}+e^{-\rho} \ln \rho-\int_{\rho}^{\infty} e^{-\tau} \ln \tau \mathrm{d} \tau
$$

and the last integral would now be convergent even if $\rho=0$, when the answer is minus Euler's constant, $\gamma$. So we can move the integration:

$$
I(\rho)=\frac{e^{-\rho}}{\rho}+e^{-\rho} \ln \rho+\gamma+\int_{0}^{\rho} e^{-\tau} \ln \tau \mathrm{d} \tau
$$

Now the last integral is better behaved: $\tau$ is small throughout and the integral is uniformly convergent. That gives us the freedom to expand the exponential as a power series, and to interchange the order of integration and summation:

$$
\int_{0}^{\rho} e^{-\tau} \ln \tau \mathrm{d} \tau=\int_{0}^{\rho} \ln \tau \sum_{n=0}^{\infty} \frac{(-\tau)^{n}}{n!} \mathrm{d} \tau=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \int_{0}^{\rho} \tau^{n} \ln \tau \mathrm{~d} \tau
$$

We can integrate by parts again, but this time differentiating the log:

$$
\int_{0}^{\rho} e^{-\tau} \ln \tau \mathrm{d} \tau=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}\left(\left[\ln \tau \frac{\tau^{n+1}}{(n+1)}\right]_{0}^{\rho}-\int_{0}^{\rho} \frac{\tau^{n}}{(n+1)} \mathrm{d} \tau\right)
$$

The final integration step gives

$$
\begin{aligned}
\int_{0}^{\rho} e^{-\tau} \ln \tau \mathrm{d} \tau & =-\sum_{n=0}^{\infty} \frac{(-\rho)^{n+1}}{(n+1)!}\left(\ln \rho-\frac{1}{(n+1)}\right) \\
& =\ln \rho\left(1-e^{-\rho}\right)+\sum_{n=0}^{\infty} \frac{(-\rho)^{n+1}}{(n+1)!(n+1)}
\end{aligned}
$$

and so

$$
I(\rho)=\frac{e^{-\rho}}{\rho}+\ln \rho+\gamma+\sum_{n=0}^{\infty} \frac{(-\rho)^{n+1}}{(n+1)!(n+1)}
$$

Thus expanding the exponential and the sum as power series in $\rho$ we obtain the asymptotic series we used earlier:

$$
I(\rho)=\frac{1}{\rho}+\ln \rho+\gamma-1-\frac{\rho}{2}+\frac{\rho^{2}}{12}-\frac{\rho^{3}}{72}+O\left(\rho^{4}\right) .
$$

### 9.2 Small parameter in the integrand

We'll look now at a specific class of integral, which may seem rather limited: but these problems crop up in many applications, and particularly in finding the outer limit of an inner boundary-layer solution. We'll use an example.
We want to know the asymptotic behaviour as $z \rightarrow \infty$ of the real part of the integral

$$
I(z)=\int_{-\infty}^{\infty} \exp [z f(t)] g(t) \mathrm{d} t
$$

where $f(t)$ and $g(t)$ are analytic functions of $t$. Our example integral $J$ has $f(t)=-t^{2}+2 i t+1$ and $g(t)=1$, so that

$$
\operatorname{Real}(J(z))=\int_{-\infty}^{\infty} \exp \left[-z\left(t^{2}-1\right)\right] \cos [2 z t] \mathrm{d} t
$$

The reason I've chosen these particular functions is that we can solve:

$$
J(z)=\int_{-\infty}^{\infty} \exp \left[-z(t-i)^{2}\right] \mathrm{d} t=\int_{-\infty-i}^{\infty-i} \exp \left[-z u^{2}\right] \mathrm{d} u
$$

and choosing the right integration contour in the $u$ plane:

the contribution from the contour ends is exponentially small and we have

$$
J(z)=\int_{-\infty}^{\infty} \exp \left[-z u^{2}\right] \mathrm{d} u=\left(\frac{\pi}{z}\right)^{1 / 2}
$$

How do we tackle the integral $I(z)$ in general? The obvious estimate would be to take $t_{0}$, the place on the integration contour at which $\operatorname{Real}(f(t))$ is maximum and expand about it:

$$
\begin{aligned}
I(z) & \sim \int_{-\infty}^{\infty} \exp \left[z\left(f\left(t_{0}\right)+\left(t-t_{0}\right)^{2} f^{\prime \prime}\left(t_{0}\right) / 2+\cdots\right] g(t) \mathrm{d} t\right. \\
& \sim \exp \left[z f\left(t_{0}\right)\right] g\left(t_{0}\right) \int_{-\infty}^{\infty} \exp \left[z\left(t-t_{0}\right)^{2} f^{\prime \prime}\left(t_{0}\right) / 2+\cdots\right] \mathrm{d} t \\
& \sim \exp \left[z f\left(t_{0}\right)\right] g\left(t_{0}\right)\left(-z f^{\prime \prime}\left(t_{0}\right)\right)^{-1 / 2}
\end{aligned}
$$

However, this is a massive overestimate. Look at our example. The real part of $f(t)$ is $-t^{2}+1$, which is maximal at $t=0$, giving the estimate

$$
J(z) \sim(2 z)^{-1 / 2} e^{z}
$$

which is too large by an exponential factor. The reason for this massive overestimate is that we have not allowed for the effect of $\cos [2 z t]$, which oscillates very rapidly when $z$ is large and causes a large cancellation effect. This cosine term came from the imaginary part of $f(t)$.
The technique which will save us is fundamentally based on the analyticity of $f(t)$ and $g(t)$. Because both they and the exponential function are analytic,
we can deform our integration contour in the complex plane without changing the result. We deform our contour to get the lowest possible estimate for our function: and because in doing so we will avoid these cancellations which caused our other estimates to be too large, we will attain the best estimate for the integral.
We begin by observing that because $f(z)$ is analytic, its real part satisfies Laplace's equation, and thus the real part can have no maxima or minima, but only saddle points. Suppose we contour the real part of $f(t)$ over the whole $t$-plane: (these contours are for $J(z)$ )


Then we can deform the contour so that it keeps Real $(f(t))$ as small as possible throughout: this will involve the contour passing over a saddle point, where the maximum value of $\operatorname{Real}(f(t))$ will be attained.
Now we can choose as our contour the route over the saddle point which keeps Imag $(f(t))$ constant along it, and "correct" back to the ends of our original integration contour in the far field, along lines of constant Real $(f(t))$. The contribution from the correction curves will be asymptotically negligible: not only will Real $(f(t))$ be much smaller there than near the saddle, but also $\operatorname{Imag}(f(t))$ is varying rapidly, giving the cancellation which was so troublesome earlier.
On the remaining integration contour, the imaginary part of $f(t)$ is constant so there will be no cancellation from rapid oscillations. Now it is appropriate to Taylor-expand $f(t)$ about the saddle point, and we can take as many terms as necessary to obtain our asymptotic expansion.
The contour which keeps the imaginary part of $f$ constant is the same contour which makes the real part change most rapidly: thus we have chosen the contour of steepest descent.
For our example, $f(t)=(t-i)^{2}$ which has a saddle point at $t=i$. If we set $t=x+i y$, the imaginary part of $f$ is $2 x(y-1)$ which is zero at the saddle point, so we choose a contour which keeps it zero: $y=1$. In fact this is exactly the procedure we carried out to calculate the integral - but as a method this works even when it doesn't exactly solve the problem.

## Example: Bessel function

The Bessel function $K_{\nu}(z)$ can be written

$$
K_{\nu}(z)=\frac{1}{2} \int_{-\infty}^{\infty} \exp [\nu t-z \cosh t] \mathrm{d} t
$$

We will take $z=1$, double the function and look for an expansion as $\nu \rightarrow \infty$ :

$$
I(\nu)=\int_{-\infty}^{\infty} \exp [\nu t-\cosh t] \mathrm{d} t
$$

If we generalise slightly to allow $f(t)$ to have $\nu$ as a parameter, we can write

$$
I(\nu)=\int_{-\infty}^{\infty} \exp [\nu(t-\cosh t / \nu)] \mathrm{d} t
$$

Since there is no trigonometric term, we know that the imaginary part of our function is zero along the real line: so in this case we will not need to find a new contour, just find the dominant contribution from the contour we started from. We look at the behaviour of $f(t)=t-\cosh t / \nu$. It has a maximum where

$$
\frac{\mathrm{d}}{\mathrm{~d} t}(t-\cosh t / \nu)=0 \quad \sinh t=\nu
$$

and about that point, we expand:

$$
\begin{aligned}
f(t) & \approx f(\operatorname{arcsinh} \nu)+\frac{1}{2}(t-\operatorname{arcsinh} \nu)^{2} f^{\prime \prime}(\operatorname{arcsinh} \nu)+\cdots \\
& =\operatorname{arcsinh} \nu-\frac{\left(\nu^{2}+1\right)^{1 / 2}}{\nu}-(t-\operatorname{arcsinh} \nu)^{2} \frac{\left(\nu^{2}+1\right)^{1 / 2}}{2 \nu}+\cdots
\end{aligned}
$$

The contributions to the integral become

$$
\begin{gathered}
I(\nu) \sim \int_{-\infty}^{\infty} \exp \left[\nu\left(\operatorname{arcsinh} \nu-\frac{\left(\nu^{2}+1\right)^{1 / 2}}{\nu}-(t-\operatorname{arcsinh} \nu)^{2} \frac{\left(\nu^{2}+1\right)^{1 / 2}}{2 \nu}\right)\right] \mathrm{d} t \\
I(\nu) \sim \frac{(2 \pi)^{1 / 2}}{\left(\nu^{2}+1\right)^{1 / 4}} \exp \left[\nu \operatorname{arcsinh} \nu-\left(\nu^{2}+1\right)^{1 / 2}\right]
\end{gathered}
$$

## Steepest descents in practice

Although we've used a contour which keeps the imaginary part of $f$ constant, and thereby makes the real part decrease as fast as possible away from the contour, this level of detail is not necessary for the leading-order term of an asymptotic expansion. In practice, any contour which crosses the saddle and descends on both sides of it will do a decent job. Numerically, it is often less expensive to carry out the integration with smaller step sizes to catch the oscillation terms than it is to find the steepest descent contour exactly. So the key point is to find the saddle (or the highest saddle) over which your contour must pass: beyond that point you have much more flexibility.

