## 6 Matching: Boundary Layers

Consider the following equation (rather similar to the example we used in section 5.1):

$$
\varepsilon \frac{\mathrm{d}^{2} f}{\mathrm{~d} x^{2}}+\frac{\mathrm{d} f}{\mathrm{~d} x}+f=0
$$

There are two solutions. One is regular:

$$
f=f_{0}(x)+\varepsilon f_{1}(x)+\cdots
$$

Substituting gives, at order 1,

$$
f_{0}^{\prime}+f_{0}=0 \Longrightarrow f_{0}=a_{0} e^{-x}
$$

At order $\varepsilon$ we have

$$
f_{1}^{\prime}+f_{1}+f_{0}^{\prime \prime}=0 \Longrightarrow f_{1}=\left[a_{1}-a_{0} x\right] e^{-x} .
$$

The second solution is singular, and the distinguished scaling (to balance the first two terms) is $\delta=\varepsilon$. We introduce a new variable $z=(x-a) / \varepsilon$ to have

$$
\frac{\mathrm{d}^{2} f}{\mathrm{~d} z^{2}}+\frac{\mathrm{d} f}{\mathrm{~d} z}+\varepsilon f=0
$$

with solution

$$
f=F_{0}(z)+\varepsilon F_{1}(z)+\cdots
$$

At order 1 we have

$$
\begin{gathered}
F_{0}^{\prime \prime}+F_{0}^{\prime}=0 \Longrightarrow F_{0}^{\prime}=-B_{0} e^{-z} \\
F_{0}(z)=A_{0}+B_{0} e^{-z} .
\end{gathered}
$$

At order $\varepsilon$ we have

$$
\begin{gathered}
F_{1}^{\prime \prime}+F_{1}^{\prime}+F_{0}=0 \Longrightarrow F_{1}^{\prime}=-A_{0}-B_{0} z e^{-z}-B_{1} e^{-z} \\
F_{1}=A_{1}-A_{0} z+B_{0}\left[z e^{-z}+e^{-z}\right]+B_{1} e^{-z} .
\end{gathered}
$$

We now have two possible solutions:

$$
\begin{aligned}
f(x) & \sim a_{0} e^{-x}+\varepsilon\left[a_{1}-a_{0} x\right] e^{-x}+\cdots \\
F(z) & \sim A_{0}+B_{0} e^{-z}+\varepsilon\left[A_{1}-A_{0} z+B_{0}\left(z e^{-z}+e^{-z}\right)+B_{1} e^{-z}\right]+\cdots
\end{aligned}
$$

Question: Will we ever need to use both of these in the same problem?
Answer: The short answer is yes. This is a second-order differential equation, so we are entitled to demand that the solution satisfies two boundary conditions.
Suppose, with the differential equation above, the boundary conditions are

$$
f=e^{-1} \text { at } x=1 \quad \text { and } \quad \frac{\mathrm{d} f}{\mathrm{~d} x}=0 \text { at } x=0 .
$$

We will start by assuming that the unstretched form will do, and apply the boundary condition at $x=1$ to it:

$$
f(x) \sim a_{0} e^{-x}+\varepsilon\left[a_{1}-a_{0} x\right] e^{-x}+\cdots
$$

$$
e^{-1}=a_{0} e^{-1}+\varepsilon\left[a_{1}-a_{0}\right] e^{-1}+\cdots
$$

which immediately yields the conditions $a_{0}=1, a_{1}=1$. If we had continued to higher orders we would be able to find the constants there as well.
Now what about the other boundary condition? We have no more disposable constants so we'd be very lucky if it worked! In fact we have

$$
f^{\prime}(x)=-a_{0} e^{-x}+\varepsilon\left[-a_{1}-a_{0}+a_{0} x\right] e^{-x}+\cdots
$$

so at $x=0$,

$$
f^{\prime}(0)=-1-2 \varepsilon+\cdots
$$

This is where we have to use the other solution. If we fix $a=0$ in the scaling for $z$, then the strained region is near $x=0$. We can re-express the boundary condition in terms of $z$ :

$$
\frac{\mathrm{d} f}{\mathrm{~d} z}=0 \text { at } z=0
$$

Now applying this boundary condition to our strained expansion gives:

$$
\begin{gathered}
F(z) \sim A_{0}+B_{0} e^{-z}+\varepsilon\left[A_{1}-A_{0} z+B_{0}\left(z e^{-z}+e^{-z}\right)+B_{1} e^{-z}\right]+\cdots \\
F^{\prime}(z)=-B_{0} e^{-z}+\varepsilon\left[-A_{0}-B_{0} z e^{-z}-B_{1} e^{-z}\right]+\cdots
\end{gathered}
$$

and at $z=0$,

$$
F^{\prime}(0)=-B_{0}+\varepsilon\left[-A_{0}-B_{1}\right]+\cdots
$$

Imposing $F^{\prime}(0)=0$ fixes $B_{0}=0, B_{1}=-A_{0}$ but does not determine $A_{0}, B_{1}$ or $A_{1}$. The solution which matches the $x=0$ boundary condition is

$$
F(z) \sim A_{0}+\varepsilon\left[A_{1}-A_{0} z-A_{0} e^{-z}\right]+\cdots
$$

We now have two perturbation expansions, one valid at $x=1$ and for most of our region, the other valid near $x=0$. We have not determined all our parameters. How will we do this? The answer is matching.

### 6.1 Intermediate variable

Suppose (as in the example above) we have two asymptotic solutions to a given problem.

- One scales normally and satisfies a boundary condition somewhere away from the tricky region: we will call this the outer solution.
- The other is expressed in terms of a scaled variable, and is valid in a narrow region, (probably) near the other boundary. We will call this the inner solution.

In order to make sure that these two expressions both belong to the same real (physical) solution to the problem, we need to match them.
In the case where the outer solution is

$$
f(x)=f_{0}(x)+\varepsilon f_{1}(x)+\varepsilon^{2} f_{2}(x)+\cdots
$$

and the inner

$$
F(z)=F_{0}(z)+\varepsilon F_{1}(z)+\varepsilon^{2} F_{2}(z)+\cdots
$$

with scalings $z=x / \varepsilon$, we will match the two expressions using an intermediate variable. This is a new variable, $\xi$, intermediate in size between $x$ and $z$, so that when $\xi$ is order $1, x$ is small and $z$ is large. We can define it as

$$
x=\varepsilon^{\alpha} \xi \Longrightarrow z=\varepsilon^{\alpha-1} \xi
$$

for $\alpha$ between 0 and 1 . It is best to keep $\alpha$ symbolic $^{5}$.
The procedure is to substitute $\xi$ into both $f(x)$ and $F(z)$ and then collect orders of $\varepsilon$ and force the two expressions to be equal. This is best seen by revisiting the previous example.

## Example continued

We had

$$
f(x)=e^{-x}+\varepsilon(1-x) e^{-x}+\cdots
$$

and

$$
F(z)=A_{0}+\varepsilon\left[A_{1}-A_{0} z-A_{0} e^{-z}\right]+\cdots
$$

with $z=x / \varepsilon$. Defining $x=\varepsilon^{\alpha} \xi$, we look first at $f(x)$ :

$$
f(x)=e^{-\varepsilon^{\alpha} \xi}+\varepsilon\left(1-\varepsilon^{\alpha} \xi\right) e^{-\varepsilon^{\alpha} \xi}+\cdots
$$

Since $\varepsilon^{\alpha} \ll 1$ we can expand the exponential terms to give

$$
f(x)=1-\varepsilon^{\alpha} \xi-\frac{1}{2} \varepsilon^{2 \alpha} \xi^{2}+\varepsilon-2 \varepsilon^{\alpha+1} \xi+O\left(\varepsilon^{2}, \varepsilon^{1+2 \alpha}, \varepsilon^{3 \alpha}\right)
$$

Now we look at $F(z)$. Note that $z=\varepsilon^{\alpha-1} \xi$, which is large.

$$
F(z)=A_{0}+\varepsilon\left[A_{1}-A_{0} \varepsilon^{\alpha-1} \xi-A_{0} e^{-\varepsilon^{\alpha-1} \xi}\right]+\cdots
$$

Here the exponential terms become very small indeed so we neglect them and have

$$
F(z)=A_{0}-A_{0} \varepsilon^{\alpha} \xi+\varepsilon A_{1}+\cdots
$$

Comparing terms of the two expansions, at order 1 we have

$$
1=A_{0}
$$

and at order $\varepsilon^{\alpha}$,

$$
-\xi=-A_{0} \xi
$$

which is automatically satisfied if $A_{0}=1$. If we fix $\alpha>1 / 2$ then the next term is order $\varepsilon$, giving

$$
1=A_{1} .
$$

The next term in the outer expansion is order $\varepsilon^{2 \alpha}$, but to match that we would have to go to order $\varepsilon^{2}$ in the inner expansion.

[^0]We have now determined all the constants to this order: so in the outer we have

$$
f(x)=e^{-x}+\varepsilon(1-x) e^{-x}+\cdots
$$

and in the inner $x=\varepsilon z$,

$$
F(z)=1+\varepsilon\left[1-z-e^{-z}\right]+\cdots
$$

Note: The beauty of the intermediate variable method for matching is that it has so much structure. If you have made any mistakes in solving either inner or outer equation, or if (by chance) you have put the inner region next to the wrong boundary, the structure of the two solutions won't match and you will know something is wrong!

### 6.2 Where is the boundary layer?

In the last example we assumed the boundary layer would be next to the lower boundary.
If we didn't know, how would we work it out?
Let's start by trying the previous example, but attempting to put the boundary layer near $x=1$.
Recall we had an outer solution:

$$
f(x) \sim a_{0} e^{-x}+\varepsilon\left[a_{1}-a_{0} x\right] e^{-x}+\cdots
$$

and an inner solution

$$
F(z) \sim A_{0}+B_{0} e^{-z}+\varepsilon\left[A_{1}-A_{0} z+B_{0}\left(z e^{-z}+e^{-z}\right)+B_{1} e^{-z}\right]+\cdots
$$

with $z=(x-a) / \varepsilon$.
This time we will try to fit the outer solution to the boundary condition at $x=0$. We have

$$
\frac{\mathrm{d} f}{\mathrm{~d} x} \sim-a_{0} e^{-x}+\varepsilon\left[a_{0} x-a_{0}-a_{1}\right] e^{-x}+\cdots
$$

so the condition is

$$
\begin{aligned}
& \frac{\mathrm{d} f}{\mathrm{~d} x}=0 \quad \text { at } \quad x=0 \\
& 0=-a_{0}+\varepsilon\left[-a_{1}-a_{0}\right]+\cdots
\end{aligned}
$$

which gives $a_{0}=0, a_{1}=0$ and so on. It is clear that we're not going to get a solution this way!
However, there is another problem, which appears when we try to fit the inner solution at the other boundary. We are setting $a=1$ and trying to fit $F(z)=$ $e^{-1}$ at $z=0$. This gives:

$$
e^{-1}=A_{0}+B_{0}+\varepsilon\left[A_{1}+B_{0}+B_{1}\right]+\cdots
$$

so $A_{0}=e^{-1}-B_{0}$ and $A_{1}=-B_{0}-B_{1}$. This seems fine, but look at the solution we get:
$F(z) \sim e^{-1}+B_{0}\left(e^{-z}-1\right)+\varepsilon\left[-e^{-1} z+B_{0}\left(z-1+(z+1) e^{-z}\right)+B_{1}\left(e^{-z}-1\right)\right]+\cdots$

Remember that, now the boundary layer is at the top, the outer limit of the inner solution will be for large negative $z$ : in other words, all of these exponentials will be growing! This can never match onto a well-behaved outer solution.
Key fact: The boundary layer is always positioned so that any exponentials in the inner solution decay as you move towards the outer.

### 6.3 Linear example

This comes from Hinch (and originally, Friedricks). Consider:

$$
\varepsilon \frac{\mathrm{d}^{2} f}{\mathrm{~d} x^{2}}+\frac{\mathrm{d} f}{\mathrm{~d} x}=1+2 x \quad \text { in } 0<x<1
$$

with boundary conditions $f(0)=0$ and $f(1)=1$.
First we look for stretches that work (note that because the equation is linear, there is no mileage in scaling $f$ ). The right hand side of the equation is always strictly order 1 in our range of $x$, so if we stretch $x$ as $x=a+\delta X$ we have three terms to compare:
$[\mathbf{A}] \varepsilon \delta^{-2}$
$[\mathbf{B}] \delta^{-1}$
[C] 1.

For very small $\delta$ we have $[\mathbf{A}] \gg[\mathbf{B}] \gg[\mathbf{C}]$, and $[\mathbf{B}]$ catches $[\mathbf{A}]$ when $\delta=\varepsilon$. Then $[\mathbf{C}]$ catches $[\mathbf{B}]$ at $\delta=1$ (which is the largest value of $\delta$ we can use, given that the range of $x$ is only order 1 ).
Thus there are two distnguished stretches: the original variable $x$ and a stretched variable $x=a+\varepsilon z$. Let us look at the regular, outer, solution first. Since we don't yet know where to put the boundary layer we won't use any boundary conditions on it yet.
We pose an expansion

$$
f \sim f_{0}+\varepsilon f_{1}+\varepsilon^{2} f_{2}+\cdots
$$

and have

$$
\begin{aligned}
f_{0}^{\prime} & =1+2 x \\
\varepsilon f_{0}^{\prime \prime}+\varepsilon f_{1}^{\prime} & =0 \\
\varepsilon^{2} f_{1}^{\prime \prime}+\varepsilon^{2} f_{2}^{\prime} & =0
\end{aligned}
$$

Integrating these in turn gives:
Order $1 f_{0}^{\prime}=1+2 x$ so $f_{0}=x+x^{2}+a_{0}$.
Order $\varepsilon f_{1}^{\prime}=-2$ so $f_{1}=-2 x+a_{1}$.
Order $\varepsilon^{2} f_{2}^{\prime}=0$ so $f_{2}=a_{2}$.
Our regular expansion is

$$
f \sim a_{0}+x+x^{2}+\varepsilon\left(a_{1}-2 x\right)+\varepsilon^{2} a_{2}+\cdots
$$

Now we move onto the inner, stretched solution. Recasting the ODE in terms of $z$ gives

$$
\frac{\mathrm{d}^{2} f}{\mathrm{~d} z^{2}}+\frac{\mathrm{d} f}{\mathrm{~d} z}=\varepsilon(1+2 a)+2 \varepsilon^{2} z
$$

We pose our expansion:

$$
f \sim F_{0}+\varepsilon F_{1}+\varepsilon^{2} F_{2}+\cdots
$$

to have

$$
\begin{aligned}
F_{0}^{\prime \prime}+F_{0}^{\prime} & =0 \\
\varepsilon F_{1}^{\prime \prime}+\varepsilon F_{1}^{\prime} & =\varepsilon(1+2 a) \\
\varepsilon^{2} F_{2}^{\prime \prime}+\varepsilon^{2} F_{2}^{\prime} & =2 \varepsilon^{2} z
\end{aligned}
$$

Solving the leading-order equation gives

$$
F_{0}^{\prime \prime}+F_{0}^{\prime}=0 \quad F_{0}=A_{0}+B_{0} e^{-z}
$$

Immediately our condition that any exponentials must decay outside the boundary layer tells us that the boundary layer is positioned near $x=0$ (so that $z$ is positive towards the interior of the domain) and thus $a=0$. That means that we can apply to our inner solution the boundary condition $f(0)=0$.

Order 1 We know $F_{0}=A_{0}+B_{0} e^{-z}$, and applying the boundary condition gives $F_{0}=A_{0}\left(1-e^{-z}\right)$.

Order $\varepsilon F_{1}^{\prime \prime}+F_{1}^{\prime}=1$ gives $F_{1}=A_{1}+B_{1} e^{-z}+z$, and the boundary condition forces $F_{1}=A_{1}\left(1-e^{-z}\right)+z$.

Order $\varepsilon^{2} F_{2}^{\prime \prime}+F_{2}^{\prime}=2 z$ gives $F_{2}=A_{2}+B_{2} e^{-z}+z^{2}-2 z$, and the boundary condition forces $F_{2}=A_{2}\left(1-e^{-z}\right)+z^{2}-2 z$.

Now we return to the outer solution, to which we can now apply the other boundary condition $f(1)=1$ :

$$
1 \sim a_{0}+2+\varepsilon\left(a_{1}-2\right)+\varepsilon^{2} a_{2}+\cdots
$$

which fixes $a_{0}=-1, a_{1}=2$ and $a_{2}=0$. We now have our two expansions:

$$
\begin{gathered}
f_{\text {outer }} \sim-1+x+x^{2}+\varepsilon(2-2 x)+O\left(\varepsilon^{3}\right) \\
f_{\text {inner }} \sim A_{0}\left(1-e^{-z}\right)+\varepsilon\left[A_{1}\left(1-e^{-z}\right)+z\right]+\varepsilon^{2}\left[A_{2}\left(1-e^{-z}\right)+z^{2}-2 z\right]+\cdots
\end{gathered}
$$

linked by the variables $x=\varepsilon z$.
To match the expansions, we introduce $\eta=\varepsilon^{-\alpha} x=\varepsilon^{1-\alpha} z$ and substitute in each:

$$
\begin{aligned}
f_{\text {outer }} & =-1+\varepsilon^{\alpha} \eta+\varepsilon^{2 \alpha} \eta^{2}+2 \varepsilon-2 \varepsilon^{1+\alpha} \eta+O\left(\varepsilon^{3}\right) \\
f_{\text {inner }} & =A_{0}+\varepsilon^{\alpha} \eta+\varepsilon^{2 \alpha} \eta^{2}+\varepsilon A_{1}-2 \varepsilon^{1+\alpha} \eta+\varepsilon^{2} A_{2}+\cdots
\end{aligned}
$$

Comparing terms at each order, we can immediately see that our expansions are succeeding in that some of the terms have already matched each other. To complete the match we need $A_{0}=-1, A_{1}=2$ and $A_{2}=0$. Thus our two expansions are

$$
\begin{gathered}
f_{\text {outer }} \sim-1+x+x^{2}+\varepsilon(2-2 x)+O\left(\varepsilon^{3}\right) \\
f_{\text {inner }} \sim e^{-z}-1+\varepsilon\left[2\left(1-e^{-z}\right)+z\right]+\varepsilon^{2}\left[z^{2}-2 z\right]+\cdots
\end{gathered}
$$

linked via $x=\varepsilon z$.

Plotting these expansions for $\varepsilon=0.1, \varepsilon=0.03$ and $\varepsilon=0.01$ shows the power of the method:


Here the outer expansions are the solid curves and the inner, the dashed curves. As $\varepsilon$ gets smaller, the outer is a good approximation for a larger and larger proportion of the range, but the inner expansion is still needed near $x=0$.

### 6.4 Another Example

This is a simplified version of an advection-diffusion problem that arose in my own research ${ }^{6}$. Solve

$$
\frac{x}{y} \frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}+\frac{f}{y}-\varepsilon \nabla^{2} f=0
$$

with boundary conditions

$$
f+\varepsilon \frac{\partial f}{\partial y}=0 \quad \text { at } \quad y=1, \quad f=2 \quad \text { at } \quad y=2
$$

The boundary condition at $y=1$ corresponds to a condition of no flux of $f$ through the boundary $y=1$.

## Outer solution

We expand the PDE:

$$
\frac{x}{y} \frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}+\frac{f}{y}-\varepsilon \frac{\partial^{2} f}{\partial x^{2}}-\varepsilon \frac{\partial^{2} f}{\partial y^{2}}=0
$$

and look for the first term of an outer solution by considering the case $\varepsilon=0$ :

$$
\frac{x}{y} \frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}+\frac{f}{y}=0 \quad x \frac{\partial f}{\partial x}-y \frac{\partial f}{\partial y}+f=0
$$

Because this is a first-order PDE we can apply the method of characteristics, solving:

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=x \quad \frac{\mathrm{~d} y}{\mathrm{~d} t}=-y
$$

[^1]which gives us the parametric curves
$$
x=x_{0} e^{t} \quad y=e^{-t} \quad x=x_{0} / y
$$
along which
$$
\frac{\mathrm{d} f}{\mathrm{~d} t}=\frac{\mathrm{d} x}{\mathrm{~d} t} \frac{\partial f}{\partial x}+\frac{\mathrm{d} y}{\mathrm{~d} t} \frac{\partial f}{\partial y}=x \frac{\partial f}{\partial x}-y \frac{\partial f}{\partial y}=-f
$$
so
$$
f=A\left(x_{0}\right) e^{-t}=A(x y) y
$$

We'll stay at one term for the outer solution.

## Scaling for the inner

We're expecting a boundary layer somewhere, because all the highest derivatives were neglected when we put $\varepsilon=0$. In fact the type of the boundary condition at $y=1$ gives us the hint: if $f+\varepsilon \partial f / \partial y=0$ then $\partial f / \partial y$ must be an order of magnitude larger than $f$. So we look to scale $y$ rather than $x$. (Another motivation for this choice is that, usually, boundary layers live near boundaries: and there are no boundaries on $x$.)
So we set $y=a+\varepsilon^{b} z$ and substitute in to the PDE:

$$
\frac{x}{\left[a+O\left(\varepsilon^{b}\right)\right]} \frac{\partial f}{\partial x}-\varepsilon^{-b} \frac{\partial f}{\partial z}+\frac{f}{\left[a+O\left(\varepsilon^{b}\right)\right]}-\varepsilon \frac{\partial^{2} f}{\partial x^{2}}-\varepsilon^{1-2 b} \frac{\partial^{2} f}{\partial z^{2}}=0
$$

Clearly the two terms which may be larger than $O(1)$ if $b>0$ are the second and last terms: $\varepsilon^{-b}$ and $\varepsilon^{1-2 b}$. Balancing the two fixes $b=1$ (which we expected from the boundary condition). Thus:

$$
-\frac{\partial f}{\partial z}-\frac{\partial^{2} f}{\partial z^{2}}+\varepsilon \frac{x}{a} \frac{\partial f}{\partial x}+\varepsilon \frac{f}{a}=O\left(\varepsilon^{2}\right) .
$$

Let's just look at the leading-order term first: $f=f_{0}+\varepsilon f_{1}+\cdots$ gives

$$
-\frac{\partial f_{0}}{\partial z}-\frac{\partial^{2} f_{0}}{\partial z^{2}}=0 \quad f_{0}=A_{0}(x)+B_{0}(x) e^{-z}
$$

The exponential in $z$ tells us that the boundary layer must be located at the lower boundary so $a=1$ and $y=1+\varepsilon z$. Then we expect the outer to satisfy the upper boundary condition at $y=2$; now we can return to the outer and complete it.

## Full outer solution

We now have the outer solution

$$
f=A(x y) y+\varepsilon f_{1}(x, y)+\cdots
$$

which we need to satisfy the boundary condition $f(x, 2)=2$. Applying this at leading order gives

$$
2=2 A(2 x) \quad A(\eta)=1 \quad f=y+\varepsilon f_{1}(x, y)+\cdots
$$

Now we can continue with the expansion: the original equation was

$$
x \frac{\partial f}{\partial x}-y \frac{\partial f}{\partial y}+f-y \varepsilon \frac{\partial^{2} f}{\partial x^{2}}-y \varepsilon \frac{\partial^{2} f}{\partial y^{2}}=0
$$

so we have

$$
\begin{aligned}
& x \partial f_{0} / \partial x-y \partial f_{0} / \partial y+f \\
& x \partial f_{1} / \partial x-y \partial f_{1} / \partial y+f_{1}-y \partial^{2} f_{0} / \partial x^{2}-y \partial^{2} f_{0} / \partial y^{2}=0 \\
& x \partial f_{2} / \partial x-y \partial f_{2} / \partial y+f_{2}-y \partial^{2} f_{1} / \partial x^{2}-y \partial^{2} f_{1} / \partial y^{2}=0
\end{aligned}
$$

with boundary conditions

$$
f_{0}(x, 2)=2 \quad f_{1}(x, 2)=0 \quad f_{2}(x, 2)=0
$$

At order 1 we know this is satisfied by $f_{0}=y$. At order $\varepsilon$ we have

$$
x \frac{\partial f_{1}}{\partial x}-y \frac{\partial f_{1}}{\partial y}+f_{1}=0
$$

which is the same equation we had for $f_{0}$, so has solution $f_{1}=A_{1}(x y) y$. This time the boundary condition gives $f_{1}=0$. We can see that this pattern will continue, and in fact $f_{n}=0$ for $n \geq 1$ : the full outer solution is

$$
f_{\text {outer }}=y
$$

## Full inner solution

We now return to the inner solution:

$$
\frac{\partial^{2} f}{\partial z^{2}}+\frac{\partial f}{\partial z}=\varepsilon \frac{x}{[1+\varepsilon z]} \frac{\partial f}{\partial x}+\varepsilon \frac{f}{[1+\varepsilon z]}-\varepsilon^{2} \frac{\partial^{2} f}{\partial x^{2}}
$$

Keeping terms up to order $\varepsilon$ gives

$$
\begin{gathered}
\frac{\partial^{2} f_{0}}{\partial z^{2}}+\frac{\partial f_{0}}{\partial z}=0 \\
\frac{\partial^{2} f_{1}}{\partial z^{2}}+\frac{\partial f_{1}}{\partial z}=x \frac{\partial f_{0}}{\partial x}+f_{0}
\end{gathered}
$$

with boundary conditions (true at each order)

$$
f+\partial f / \partial z=0 \quad \text { at } z=0
$$

At order 1 we have

$$
f_{0}=A_{0}(x)+B_{0}(x) e^{-z}
$$

and the boundary condition gives $A_{0}(x)=0: f_{0}=B_{0}(x) e^{-z}$.
At order $\varepsilon$ we have

$$
\frac{\partial^{2} f_{1}}{\partial z^{2}}+\frac{\partial f_{1}}{\partial z}=\left[x B_{0}^{\prime}+B_{0}\right] e^{-z}
$$

which gives

$$
f_{1}=A_{1}(x)+B_{1}(x) e^{-z}-\left[x B_{0}^{\prime}+B_{0}\right] z e^{-z} .
$$

Applying the boundary condition fixes $A_{1}(x)=\left[x B_{0}^{\prime}+B_{0}\right]$. Thus our solution (to order $\varepsilon$ ) is

$$
f=B_{0}(x) e^{-z}+\varepsilon\left[\left(x B_{0}^{\prime}(x)+B_{0}(x)\right)\left(1-z e^{-z}\right)+B_{1}(x) e^{-z}\right]+O\left(\varepsilon^{2}\right) .
$$

## Matching

Our two solutions are:

$$
f_{\text {outer }}=y
$$

$$
f_{\text {inner }}=B_{0}(x) e^{-z}+\varepsilon\left[\left(x B_{0}^{\prime}(x)+B_{0}(x)\right)\left(1-z e^{-z}\right)+B_{1}(x) e^{-z}\right]+O\left(\varepsilon^{2}\right) .
$$

Using an intermediate variable $y=1+\varepsilon^{\alpha} \eta, z=\varepsilon^{\alpha-1} \eta$, the outer becomes

$$
f_{\text {outer }}=1+\varepsilon^{\alpha} \eta
$$

and the inner (neglecting decaying exponentials)

$$
f_{\text {inner }}=\varepsilon\left(x B_{0}^{\prime}(x)+B_{0}(x)\right)+O\left(\varepsilon^{2}\right) .
$$

There is nothing in the inner large enough to match onto the 1 in the outer. However, remember we started from a linear equation. Along with the fact that the inner boundary condition was homogeneous, that means that if $f_{\text {inner }}$ is a solution, so is $\varepsilon^{-1} f_{\text {inner }}$. So we try that:

$$
\begin{aligned}
f_{\text {inner,new }} & =\varepsilon^{-1} B_{0}(x) e^{-z}+\left(x B_{0}^{\prime}(x)+B_{0}(x)\right)\left(1-z e^{-z}\right)+B_{1}(x) e^{-z}+O(\varepsilon) \\
& \sim\left(x B_{0}^{\prime}(x)+B_{0}(x)\right)+O(\varepsilon) \text { as } z \rightarrow \infty
\end{aligned}
$$

Now we can match the two functions if

$$
x B_{0}^{\prime}(x)+B_{0}(x)=1
$$

which is just an ODE. Solving gives $B_{0}(x)=1+C / x$ and since the line $x=0$ is within our domain, we require $C=0$ for regularity. Thus:

$$
\begin{aligned}
f_{\text {outer }} & =y \\
f_{\text {inner }} & =\varepsilon^{-1} e^{-z}+\left(1-z e^{-z}\right)+B_{1}(x) e^{-z}+O(\varepsilon)
\end{aligned}
$$

with $y=1+\varepsilon z$. To determine $B_{1}$ we would have to calculate the $f_{2}$ term of the inner expansion.


[^0]:    ${ }^{5}$ However, occasionally you may find it quicker to pick a value of $\alpha=1 / 2$, say. Be warned: sometimes there is only a specific range of $\alpha$ which works.

[^1]:    ${ }^{6}$ JFM, 534, 97-114, 2005

