## 5 Scalings with differential equations

### 5.1 Stretched coordinates

Consider the first-order linear differential equation

$$
\varepsilon \frac{\mathrm{d} f}{\mathrm{~d} x}+f=0
$$

Since it is first order, we expect a single solution to the homogeneous equation. If we try our standard method and set $\varepsilon=0$ we get $f=0$ which is clearly not a good first term of an expansion!
Solving the differential equation directly gives

$$
f=A_{0} \exp [-x / \varepsilon]
$$

This gives us the clue that what we should have done was change to a stretched variable $z=x / \varepsilon$.
Let us ignore the full solution and simply make that substitution in our governing equation. Note that $\mathrm{d} f / \mathrm{d} x=\mathrm{d} f / \mathrm{d} z \mathrm{~d} z / \mathrm{d} x=\varepsilon^{-1} \mathrm{~d} f / \mathrm{d} z$.

$$
\varepsilon \varepsilon^{-1} \frac{\mathrm{~d} f}{\mathrm{~d} z}+f=0 \quad \frac{\mathrm{~d} f}{\mathrm{~d} z}+f=0
$$

Now the two terms balance: that is, they are the same order in $\varepsilon$. Clearly the solution to this equation is now $A_{0} \exp [-z]$ and we have found the result.
This is a general principle. For a polynomial, we look for a distinguished scaling of the quantity we are trying to find. For a differential equation, we look for a stretched version of the independent variable.
The process is very similar to that for a polynomial. We use a trial scaling $\delta$ and set

$$
x=a+\delta(\varepsilon) X
$$

Then we vary $\delta$, looking for values at which the two largest terms in the scaled equation balance.

Let's work through the process for the following equation:

$$
\varepsilon \frac{\mathrm{d}^{2} f}{\mathrm{~d} x^{2}}+\frac{\mathrm{d} f}{\mathrm{~d} x}-f=0
$$

Again, we note that if $x=a+\delta X$ then $\mathrm{d} / \mathrm{d} x=\mathrm{d} / \mathrm{d} X \mathrm{~d} X / \mathrm{d} x=\delta^{-1} \mathrm{~d} / \mathrm{d} X$. We substitute in these scalings, and then look at gradually increasing $\delta$ :
$[\mathbf{A}] \varepsilon \delta^{-2}$
[B] $\delta^{-1}$
[C] 1

For small $\delta$ term $[\mathbf{A}]$ is the largest; as $\delta$ increases term $[\mathbf{B}]$ catches up first at $\delta=\varepsilon$. Then [C] catches [B] at $\delta=1$ so the two distinguished stretches are $\delta=\varepsilon$ and $\delta=1$.
For $\delta=1$ we can treat this as a regular perturbation expansion:

$$
f=f_{0}(x)+\varepsilon f_{1}(x)+\cdots
$$

$$
\begin{array}{r}
f_{0}^{\prime}-f_{0}=0 \\
\varepsilon f_{0}^{\prime \prime}+\varepsilon f_{1}^{\prime}-f_{1}=0
\end{array}
$$

At leading order we have

$$
f_{0}^{\prime}-f_{0}=0 \quad f_{0}(x)=a_{0} e^{x}
$$

and the next order becomes

$$
f_{1}^{\prime}-f_{1}=-a_{0} e^{x} \quad f_{1}(x)=a_{1} e^{x}-a_{0} x e^{x}
$$

so the regular solution begins

$$
f(x) \sim a_{0} e^{x}+\varepsilon\left(a_{1}-a_{0} x\right) e^{x}+\cdots
$$

For $\delta=\varepsilon$ we use our new variable $X=\varepsilon^{-1}(x-a)$ and work with the new governing equation:

$$
\frac{\mathrm{d}^{2} f}{\mathrm{~d} X^{2}}+\frac{\mathrm{d} f}{\mathrm{~d} X}-\varepsilon f=0
$$

Again, with the new scaling, we try a regular perturbation expansion:

$$
f=f_{0}+\varepsilon f_{1}+\varepsilon^{2} f_{2}+\cdots
$$

We substitute this in and collect powers of $\varepsilon$ :

$$
\begin{aligned}
f_{0 X X}+f_{0 X} & =0 \\
\varepsilon f_{1 X X}+\varepsilon f_{1 X}-\varepsilon f_{0} & =0 \\
\varepsilon^{2} f_{2 X X}+\varepsilon^{2} f_{2 X}-\varepsilon^{2} f_{1} & =0
\end{aligned}
$$

We then solve at each order:

$$
\begin{array}{lll}
\varepsilon^{0}: f_{0 X X}+f_{0 X}=0 & f_{0}=A_{0}+B_{0} e^{-X} \\
\varepsilon^{1}: & f_{1 X X}+f_{1 X}-f_{0}=0 & f_{1}=A_{0} X-B_{0} X e^{-X}+A_{1}+B_{1} e^{-X}
\end{array}
$$

and so on. Of course, without boundary conditions to apply, this process spawns large numbers of unknown constants. Rescaling to our original variable completes the process:

$$
\begin{aligned}
f(x) & \sim A_{0}+B_{0} \exp \left[-\frac{(x-a)}{\varepsilon}\right] \\
& +\varepsilon\left\{A_{1}+A_{0}\left(\frac{x-a}{\varepsilon}\right)+\left(B_{1}-B_{0}\left(\frac{x-a}{\varepsilon}\right)\right) \exp \left[-\frac{(x-a)}{\varepsilon}\right]\right\}+\cdots
\end{aligned}
$$

Note that this expansion is only valid where $X=(x-a) / \varepsilon$ is order 1: that is, for $x$ close to the (unknown) value $a$.

### 5.2 Must two terms dominate?

In fact we've been rather harsh in our conditions. To find all roots of a polynomial, we only ever consider scalings where the two largest terms balance. But for a differential equation we can, if we like, be more relaxed. We must include at least one scaling in which the highest-order derivative participates, otherwise we have lost one solution of our equation; but it is possible to have a solution in which a derivative (usually the highest derivative) dominates alone. Sometimes this is a (non-fatal) sign that we could have chosen our scaling better; sometimes, in complicated systems, it's unavoidable.

## Example

$$
\frac{\mathrm{d} f}{\mathrm{~d} x}+\varepsilon f=0 \quad \text { with boundary condition } f(0)=C
$$

Of course in this case we can either find the scaling instantly ( $x \sim \varepsilon^{-1}$ ) or solve the whole equation. But suppose instead we were to try a regular expansion:

$$
\begin{aligned}
& f=f_{0}+\varepsilon f_{1}+\varepsilon^{2} f_{2}+\varepsilon^{3} f_{3}+\cdots \\
f_{0}^{\prime} & +\varepsilon f_{1}^{\prime}+\varepsilon^{2} f_{2}^{\prime}+\varepsilon^{3} f_{3}^{\prime}+\cdots \\
& +\varepsilon f_{0}+\varepsilon^{2} f_{1}+\varepsilon^{3} f_{2}=0
\end{aligned}
$$

then solving at each order in turn, applying the boundary condition, gives

$$
\begin{array}{ccc}
f_{0}^{\prime}=0 & f_{0}=a_{0} & f_{0}=C \\
f_{1}^{\prime}+C=0 & f_{1}=a_{1}-C x & f_{1}=-C x \\
f_{2}^{\prime}-C x=0 & f_{2}=a_{2}+\frac{1}{2} C x^{2} & f_{2}=\frac{1}{2} C x^{2}
\end{array}
$$

which is a perfectly good regular expansion for the true solution:

$$
f=C\left\{1+\varepsilon x+\frac{1}{2} \varepsilon x^{2}+\cdots\right\} \quad f=C \exp \varepsilon x
$$

### 5.3 Nonlinear differential equations: scale and stretch

Recall that for a linear differential equation, if $f$ is a solution then so is $C f$ for any constant $C$. So if $f(x ; \varepsilon)$ is a solution as an asymptotic expansion, then $C f$ is a valid asymptotic solution even if $C$ is an arbitrary function of $\varepsilon$.
The same is not true of nonlinear differential equations. Suppose we are looking at the equation:

$$
\frac{\mathrm{d}^{2} f}{\mathrm{~d} x^{2}}+\varepsilon f(x) \frac{\mathrm{d} f}{\mathrm{~d} x}+f^{2}(x)=0
$$

There are two different types of scaling we can apply: we can scale $f$, or we can stretch $x$. To get all valid scalings we need to do both of these at once.
Let us take $f=\varepsilon^{\alpha} F$ where $F$ is strictly ord(1), and $x=a+\varepsilon^{\beta} z$ with $z$ also strictly ord(1). Then a derivative scales like $\mathrm{d} / \mathrm{d} x \sim \varepsilon^{-\beta} \mathrm{d} / \mathrm{d} z$ and we can look at the scalings of all our terms:

$$
\begin{gathered}
\frac{\mathrm{d}^{2} f}{\mathrm{~d} x^{2}}+\varepsilon f(x) \frac{\mathrm{d} f}{\mathrm{~d} x}+f^{2}(x)=0 \\
\varepsilon^{\alpha} \varepsilon^{-2 \beta} \\
\varepsilon \varepsilon^{2 \alpha} \varepsilon^{-\beta} \quad \varepsilon^{2 \alpha}
\end{gathered}
$$

As always with three terms in the equation, there are three possible balances.

- For terms I and II to balance, we need $\alpha-2 \beta=2 \alpha+1-\beta$. This gives $\alpha+\beta+1=0$, so that terms I and II scale as $\varepsilon^{2+3 \alpha}$, and term III scales as $\varepsilon^{2 \alpha}$. We need the balancing terms to dominate, so we also need $2 \alpha>2+3 \alpha$ which gives $\alpha<-2$.
- For terms I and III to balance, we need $\alpha-2 \beta=2 \alpha$. This gives $\alpha=-2 \beta$, so that terms I and III scale as $\varepsilon^{2 \alpha}$ and term II scales as $\varepsilon^{1+5 \alpha / 2}$. Again, we need the non-balancing term to be smaller than the others, so we need $1+5 \alpha / 2>2 \alpha$, i.e. $\alpha>-2$.
- Finally, to balance terms II and III, we need $2 \alpha-\beta+1=2 \alpha$ which gives $\beta=1$. Then terms II and III scale as $\varepsilon^{2 \alpha}$ and term I scales as $\varepsilon^{\alpha-2}$, so to make term I smaller than the others we need $\alpha-2>2 \alpha$, giving $\alpha<-2$.

We can plot the lines in the $\alpha-\beta$ plane where these balances occur, and in the regions between, which term (I, II or III) dominates:


We can see that there is a distinguished scaling $\alpha=-2, \beta=1$ where all three terms balance. If we apply this scaling to have $z=(x-a) / \varepsilon$ and $F=\varepsilon^{2} f$ then the governing ODE for $F(z)$ (after multiplication of the whole equation by $\varepsilon^{4}$ ) becomes

$$
\frac{\mathrm{d}^{2} F}{\mathrm{~d} z^{2}}+F \frac{\mathrm{~d} F}{\mathrm{~d} z}+F^{2}=0
$$

This is very nice: but it may not always be appropriate: the boundary conditions may fix the size of either $f$ or $x$, in which case the best you can do may be one of the simple balance points (i.e. a point $(\alpha, \beta)$ lying on one of the lines in the diagram).

### 5.4 Scale and stretch with linear differential equations

Scaling might not seem useful in a linear equation: but if the equation is expressed in terms of more than one physical variable, the relative scales of the different variables are not necessarily obvious beforehand. To give an example I'm using the ODEs which result from a particular linear stability problem I've studied ${ }^{4}$ : I've thrown away a few terms to make it less daunting, but there's still plenty to worry about!
There are 7 variables: a streamfunction $\psi$, three stress components $s_{1}, s_{2}$ and $s_{3}$, the pressure $p$, and two polymer stresses $t_{1}$ and $t_{2}$.
There are also two physical parameters: $k$ (a wavenumber) and $l$ (a diffusion lengthscale). Either of them can be small.

$$
k s_{1}+s_{2}^{\prime}=0 \quad k s_{2}+s_{3}^{\prime}=0
$$

[^0]\[

$$
\begin{aligned}
s_{1} & =-p+2 k \psi^{\prime}+t_{1} \\
s_{2} & =\psi^{\prime \prime}+t_{2} \\
s_{3} & =-p-2 k \psi^{\prime}-t_{1} \\
t_{1}-l^{2} t_{1}^{\prime \prime} & =2 k \psi^{\prime}+2\left(\psi^{\prime \prime}+k^{2} \psi\right) \\
t_{2}-l^{2} t_{2}^{\prime \prime} & =\psi^{\prime \prime}-k^{2} \psi
\end{aligned}
$$
\]

## Small $k$ : regular expansion

Physical understanding allows us to predict that the expansion for small $k$ will be regular: this is because small $k$ means we're studying long waves, and we don't expect anything to happen on a very short lengthscale for long waves.
Since the whole system is linear, there's no amplitude, so we can freely choose one variable to make strictly order 1 . Here we'll choose the streamfunction, $\psi$ :

$$
\psi \sim \psi_{0}+k \psi_{1}+\cdots
$$

Now looking at the last two equations, and assuming that $t_{1}$ and $t_{2}$ are the same size, the dominant terms on their right hand sides are $\psi^{\prime \prime}$ in both cases: so we can take $t_{1}$ and $t_{2}$ to be order 1 as well.
The tricky part comes in deciding the size of the $s_{i}$ terms and $p$. They could all be order 1 ; then the first two equations would give us, at leading order,

$$
s_{2}^{\prime}=s_{3}^{\prime}=0
$$

in fact this is a "second-best" scaling and we can do better by allowing $s_{1}, s_{3}$ and $p$ to have singular scalings:

$$
s_{1}=k^{-1} \bar{s}_{1}+O(1) \quad s_{3}=k^{-1} \bar{s}_{3}+O(1) \quad p=k^{-1} \bar{p}+O(1) .
$$

Then our set of ODEs at leading order is

$$
\begin{aligned}
\bar{s}_{1}+s_{2}^{\prime}=0 \quad \bar{s}_{3}^{\prime} & =0 \quad \bar{s}_{1}=\bar{s}_{3}=-\bar{p} \\
s_{2,0} & =\psi_{0}^{\prime \prime}+t_{2} \\
t_{1}-l^{2} t_{1}^{\prime \prime} & =2 \psi_{0}^{\prime \prime} \\
t_{2}-l^{2} t_{2}^{\prime \prime} & =\psi_{0}^{\prime \prime}
\end{aligned}
$$

## Small $l$ : regular expansion

When the lengthscale $l$ is small, there are two expansions. The first is regular and in fact does not need any scaling at all: the leading order equations are simply

$$
\begin{aligned}
k s_{1} & +s_{2}^{\prime}=0 \quad k s_{2}+s_{3}^{\prime}=0 \\
s_{1} & =-p+2 k \psi^{\prime}+t_{1} \\
s_{2} & =\psi^{\prime \prime}+t_{2} \\
s_{3} & =-p-2 k \psi^{\prime}-t_{1} \\
t_{1} & =2 k \psi^{\prime}+2\left(\psi^{\prime \prime}+k^{2} \psi\right) \\
t_{2} & =\psi^{\prime \prime}-k^{2} \psi
\end{aligned}
$$

However, we have thrown away our highest-order derivatives of $t_{1}$ and $t_{2}$ in making this expansion, so we know there must be a singular expansion as well.

## Small $l$ : stretching coordinate

Because our equations are linear, $t_{1}$ will always be larger than $l^{2} t_{1}^{\prime \prime}$ no matter how we scale $t_{1}$; in order to bring back the highest derivatives we will have to stretch the underlying coordinate.
The terms we are concerned about are $l^{2} t_{1}^{\prime \prime}$ and $l^{2} t_{2}^{\prime \prime}$, and they appear in equations with terms $t_{1}$ and $t_{2}$. Balancing these two types of term immediately suggests a stretch $x=a+l z$ (where $x$ is our original coordinate). Applying this to the original equations, and using prime now to represent derivatives w.r.t. $z$, we have

$$
\begin{aligned}
& k s_{1}+l^{-1} s_{2}^{\prime}=0 \quad k s_{2}+l^{-1} s_{3}^{\prime}=0 \\
& s_{1}=-p+2 k l^{-1} \psi^{\prime}+t_{1} \\
& s_{2}=l^{-2} \psi^{\prime \prime}+t_{2} \\
& s_{3}=-p-2 k l^{-1} \psi^{\prime}-t_{1} \\
& t_{1}-t_{1}^{\prime \prime}=2 k l^{-1} \psi^{\prime}+2\left(l^{-2} \psi^{\prime \prime}+k^{2} \psi\right) \\
& t_{2}-t_{2}^{\prime \prime}=l^{-2} \psi^{\prime \prime}-k^{2} \psi
\end{aligned}
$$

Again, we will start by fixing $\psi$ strictly order 1 : then it appears from the last two equations that $t_{1}$ and $t_{2}$ will be order $l^{-2}$ and (following through) so will all the stresses $s_{i}$. The leading-order equations (in the new scaled variables) in this case are:

$$
\begin{array}{lll}
s_{2}^{\prime}=s_{3}^{\prime}=0 & t_{1}-t_{1}^{\prime \prime}=2 \psi^{\prime \prime} & t_{2}-t_{2}^{\prime \prime}=\psi^{\prime \prime} \\
s_{1}=-p+t_{1} & s_{2}=\psi^{\prime \prime}+t_{2} & s_{3}=-p-t_{1}
\end{array}
$$

But that's not the only scaling that works.
If we continue with $\psi$ being strictly order 1 , but consider the possibility that its leading-order term is a constant, then the forcing terms in the $t$ equations are order 1 , and we can use the same trick as for the small- $k$ case to get $s_{1}$ involved in the first equation: put $p$ order $l^{-1}$, then $s_{1}$ and $s_{3}$ are also order $l^{-1}$ and the leading-order equations are:

$$
\begin{gathered}
s_{1}=s_{3}=-p \quad s_{2}=l^{-2} \psi^{\prime \prime}+t_{2} \\
k s_{1}+s_{2}^{\prime}=s_{3}^{\prime}=0 \quad t_{1}-t_{1}^{\prime \prime}=2 k^{2} \psi \quad t_{2}-t_{2}^{\prime \prime}=-k^{2} \psi
\end{gathered}
$$

This seems less obvious and perhaps even less convincing than the straighforward scaling above: but in the real problem I was solving, this scaling gave the balances we needed.


[^0]:    ${ }^{4}$ H J Wilson \& S M Fielding. J. Non-Newtonian Fluid Mech., 138, 181-196, (2006)

