4 Rescaling

In this section we'll look at one of the reasons that our $\varepsilon = 0$ system might not have enough solutions, and introduce a tool that is fundamental to all perturbation systems. We'll start with a very simple example and work up from there.

4.1 Example algebraic equation

Here our model equation is

$$\varepsilon x^2 + x - 1 = 0. \tag{2}$$

Suppose we try a regular perturbation expansion on it. Setting $\varepsilon = 0$ gives

$$x - 1 = 0,$$

with just the one solution $x = 1$. Since we started with a second-degree polynomial we know we have lost one of our solutions; however, if we carry on with the regular perturbation expansion we will get a perfectly valid series for the root near $x = 1$.

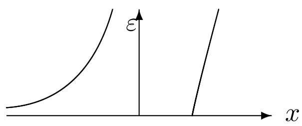
Now let us look at the true solution to see what's gone wrong.

$$x = \frac{-1 \pm \sqrt{1 + 4\varepsilon}}{2\varepsilon}$$

As $\varepsilon \to 0$, the leading-order terms of the two roots are

$$x = 1 + O(\varepsilon); \qquad \text{and} \qquad -\frac{1}{\varepsilon} + O(1).$$

The first of these is amenable to the simplistic approach; we haven't seen the second root because it $\to \infty$ as $\varepsilon \to 0$.



For this second root, let us try a series

$$x = x_{-1}\varepsilon^{-1} + x_0 + \varepsilon x_1 + \cdots$$

We substitute it into (2):

$$\begin{array}{cccccccc} & x_{-1}^2\varepsilon^{-1} & + & 2x_{-1}x_0 & + & \varepsilon(x_0^2 + 2x_{-1}x_1) & + & \cdots \\ + & x_{-1}\varepsilon^{-1} & + & x_0 & + & \varepsilon x_1 & + & \cdots \\ & & - & 1 & & & & \end{array} = 0$$

and collecting powers of ε gives:

$$\begin{array}{rcrcl} \varepsilon^{-1} & : & x_{-1}^2 + x_{-1} & = 0 & ; \quad x_{-1} = 0 \ , \ -1 \\ \varepsilon^{0} & : & 2x_{-1}x_0 + x_0 - 1 & = 0 & ; \quad x_0 = 1 \ , \ -1 \\ \varepsilon^{1} & : & x_0^2 + 2x_{-1}x_1 + x_1 & = 0 & ; \quad x_1 = -1 \ , \ 1 \end{array}$$

Note that we can now get the expansions for both of the roots using the same method.

4.2 Finding the scaling

What do we do if we can't use the exact solution to tell us about the first term in the series?

We use a trial scaling δ. We put

$$x = \delta(\varepsilon)X$$

with δ being an unknown function of ε, and X being strictly order 1. We call this $X = \mathrm{ord}(1)$: as $\varepsilon \to 0$, X is neither small nor large.

Let's try it for our example equation: $\varepsilon x^2 + x - 1 = 0$. We put in the new form:

$$\varepsilon\delta^2 X^2 + \delta X - 1 = 0$$

and then look at the different possible values of δ. We will only get an order 1 solution for X if the biggest term in the equation is the same size as another term: a **dominant balance** or **distinguished scaling**.

Finding scalings in large systems is more of an art than a science – it's easy to check your scaling works, but finding it in the first place is tricky. But with small systems, it's quite straightforward. I view this process in two ways: one completely systematic (but really only practical with a three-term equation) and the other more of a mental picture.

Systematic method

Since we need the two largest terms to balance, we try all the possible pairs of terms and find the value of δ at which they are the same size. Then for each pair we check that the other term is not bigger than our balancing size.

Balance terms 1 and 2 These two are the same size when $\varepsilon\delta^2 = \delta$ which gives $\delta = \varepsilon^{-1}$. Then both terms 1 and 2 scale as ε^{-1} and term 3 is smaller – so this scaling works.

Balance terms 1 and 3 These two balance when $\varepsilon\delta^2 = 1$ and so $\delta = \varepsilon^{-1/2}$. Then our two terms are both order 1, and term 2 scales as $\varepsilon^{-1/2}$ which is bigger. The balancing terms don't dominate so this scaling is no use.

Balance terms 2 and 3 These two balance when $\delta = 1$, when they are both order 1. Then term 1 is order ε, which is smaller: so we have a working balance at $\delta = 1$.

This process quickly gives us the only two scalings which work: $\delta = \varepsilon^{-1}$ and $\delta = 1$.

Horse-race picture

Think of the terms as horses, which "race" as we change δ. The largest term is considered to be leading, and we are interested in the moment when the lead horse is overtaken: that is, the two biggest terms are equal in size.

The three horses in our case are

$$[\mathbf{A}]\ \ \varepsilon\delta^2 \qquad [\mathbf{B}]\ \ \delta \qquad [\mathbf{C}]\ \ 1$$

and we will start from the point $\delta \approx 0$. Initially, [**C**] is ahead, with [**B**] second and [**A**] a distant third.

As we increase δ, each horse moves according to its power of δ: higher powers move faster (but start further behind). We are looking for the first moment that one of [**A**] or [**B**] catches [**C**]. A quick glance tells us that for [**B**] it will happen at $\delta = 1$ whereas for [**A**] we have to wait until $\delta > 1$. So the first balance is at $\delta = 1$, when [**B**] overtakes [**C**].

Now because [**C**] is the slowest horse (in fact stationary) it will never catch [**B**] again, so we only need to look for the moment (if any) when [**A**] overtakes [**B**]. This is given by $\varepsilon\delta^2 = \delta$ which gives our second balancing scaling of $\delta = \varepsilon^{-1}$.

4.3 Impossible equations: non-integral powers

Try this algebraic equation:

$$(1-\varepsilon)x^2 - 2x + 1 = 0.$$

Setting $\varepsilon = 0$ gives a double root $x = 1$. Now we try an expansion:

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

Substituting in gives

$$\begin{array}{rcccccccc} & 1 & + & 2\varepsilon x_1 & + & \varepsilon^2(x_1^2 + 2x_2) & + & \cdots \\ & & - & \varepsilon & - & 2\varepsilon^2 x_1 & + & \cdots \\ - & 2 & - & 2\varepsilon x_1 & - & 2\varepsilon^2 x_2 & + & \cdots \\ + & 1 & & & & & = & 0 \end{array}$$

At ε^0, as expected, the equation is automatically satisfied. However, at order ε^1, the equation is

$$2x_1 - 1 - 2x_1 = 0 \qquad 1 = 0$$

which we can never satisfy. Something has gone wrong...

In fact in this case we should have expanded in powers of $\varepsilon^{1/2}$. If we set

$$x = 1 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + \cdots$$

then we get

$$\begin{array}{rcccccccc} & 1 & + & 2\varepsilon^{1/2} x_{1/2} & + & \varepsilon(x_{1/2}^2 + 2x_1) & + & \cdots \\ & & & & - & \varepsilon & + & \cdots \\ - & 2 & - & 2\varepsilon^{1/2} x_{1/2} & - & 2\varepsilon x_1 & + & \cdots \\ + & 1 & & & & & = & 0 \end{array}$$

At order ε^0 we are still OK as before; at order $\varepsilon^{1/2}$ we have

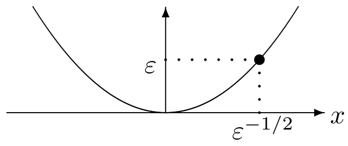
$$2x_{1/2} - 2x_{1/2} = 0$$

which is also automatically satisfied. We don't get to determine anything until we go to order ε^1, where we get

$$x_{1/2}^2 + 2x_1 - 1 - 2x_1 = 0 \qquad x_{1/2}^2 - 1 = 0$$

giving two solutions $x_{1/2} = \pm 1$. Both of these are valid and will lead to valid expansions if we continue.

We could have predicted that there would be trouble when we found the double root: near a quadratic zero of a function, a change of order $\varepsilon^{1/2}$ in x is needed to change the function value by ε:



4.4 Choosing the expansion series

In the example above, if we had begun by defining $\delta = \varepsilon^{1/2}$ we would have had a straightforward regular perturbation series in δ. But how do we go about spotting what series to use?

In practice, it is usually worth trying an obvious series like ε, ε^2, ε^3 or, if there is a distinguished scaling with fractional powers, then a power series based on that. But this trial-and-error method, while quick, is not guaranteed to succeed.

In general, for an equation in x, we can pose a series

$$x \sim x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon) + x_2\delta_2(\varepsilon) + \cdots$$

in which x_i is strictly order 1 as $\varepsilon \to 0$ (i.e. tends neither to zero nor infinity) and the series of functions $\delta_i(\varepsilon)$ has $\delta_0(\varepsilon) \gg \delta_1(\varepsilon) \gg \delta_2(\varepsilon) \cdots$ for $\varepsilon \ll 1$.

Then at each order we look for a distinguished scaling. Let us work through an example:

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\varepsilon.$$

In this case there is a solution near $x = 0$, which we will investigate.

First let us sort out the trigonometric term, expanding it as a Taylor series about $x = 0$:

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin x\cos\left(\frac{\pi}{4}\right) + \cos x\sin\left(\frac{\pi}{4}\right)\right] =$$
$$\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] = \sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

The governing equation becomes

$$-\frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6) = -\frac{1}{6}\varepsilon.$$

$$x^3 - \frac{x^4}{4} - \frac{x^5}{20} + O(x^6) = \varepsilon.$$

We pose a series

$$x = x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon) + \cdots$$

and substitute it. The leading term on the left hand side is $x_0^3\delta_0^3$, and on the right hand side is ε. So we set $\delta_0 = \varepsilon^{1/3}$ and $x_0 = 1$.

Now we have

$$x = \varepsilon^{1/3} + x_1\delta_1(\varepsilon) + \cdots$$

which we substitute into the governing equation. Remembering that $\delta_1 \ll \varepsilon^{1/3}$ and keeping terms up to order $\varepsilon^{2/3}\delta_1$ and $\varepsilon^{4/3}$ (neglecting only terms which are guaranteed to be smaller than one of these), we have

$$3x_1\varepsilon^{2/3}\delta_1 - \varepsilon^{4/3}/4 = 0.$$

To make this work, we need $\delta_1 = \varepsilon^{2/3}$ and then $x_1 = 1/12$.

The first two terms of the solution are:

$$x = \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3} + \cdots$$

4.5 A worse expansion series: Logarithms

Let us consider the equation (with $\varepsilon > 0$):

$$e^{-x} - \varepsilon x = 0.$$

We're looking for the leading-order scaling for x:

$$x \sim x_0\delta_0 + x_1\delta_1 + \cdots$$

As a quick first hack, we need to check we expect a solution at all. Both e^{-x} and $-\varepsilon x$ are decreasing functions so the whole left hand side is a decreasing function of x. At $x = 0$ the function value is 1; for large x, it is negative. Therefore we expect exactly one root, and we expect it to lie in positive x.

In order to see the scaling of the leading term, we will look at the function

$$f(x) = x^{-1}e^{-x} \qquad \text{(we need } f(x) = \varepsilon\text{)}.$$

It is also a decreasing function, moving from ∞ at $x = 0$ to 0 as $x \to \infty$. We can check the value of $f(x)$ for various values of x, so that we know where to look for the root.

If $x = 1$ then $f(x) = e^{-1}$ which is too large: so we need $x > 1$.

If $x = \varepsilon^{-1}$, then $f(x) = \varepsilon \exp(-\varepsilon^{-1})$ which is exponentially small: so we need $x < \varepsilon^{-1}$.

If $x = \varepsilon^{-\alpha}$ for some fixed positive α, then $f(x) = \varepsilon^{\alpha} \exp(-\varepsilon^{-\alpha})$ which is still exponentially small: so we need a value of x which is larger than 1 but smaller than any negative power of ε. This naturally leads us to the logarithm.

If we try $\delta_0 = \ln(1/\varepsilon)$ (with the inverse present so that δ_0 is positive, which makes everything more intuitive) then the leading-order approximation to $f(x)$ is

$$f(x_0\delta_0) = x_0^{-1}\delta_0^{-1}\exp[-\delta_0 x_0] = \frac{\varepsilon^{x_0}}{x_0 \ln(1/\varepsilon)}$$

Does this work? Let's pick values of x_0 to try.

- If $x_0 = 1$ then $f(x) = \varepsilon/\ln(1/\varepsilon) \ll \varepsilon$.
- If $x_0 = 1/2$ then $f(x) = 2\varepsilon^{1/2}/\ln(1/\varepsilon) \gg \varepsilon$.

These two order-1 values for x_0 bound our root, so we know we have found the right scaling to start with. Once we've got the first scaling it all becomes much easier.

Now let's continue with the series:

$$x = x_0 \ln(1/\varepsilon) + \delta_1 x_1 + \delta_2 x_2 + \cdots$$

Before we go any further, note that $\ln(1/\varepsilon)$ is large and positive, and let us denote

$$L_1 = \ln(1/\varepsilon), \qquad L_2 = \ln\ln(1/\varepsilon) = \ln L_1.$$

The scaling of these terms is $\varepsilon^{-\alpha} \gg L_1 \gg L_2 \gg 1$.

Now on with our expansion. We substitute the first two terms into the governing equation to have:

$$\exp(-[x_0 \ln(1/\varepsilon) + x_1\delta_1 + \cdots]) - \varepsilon[x_0 \ln(1/\varepsilon) + x_1\delta_1 + \cdots] = 0$$

$$\varepsilon^{x_0} \exp(-[x_1\delta_1 + \cdots]) = x_0 \varepsilon \ln(1/\varepsilon) + \cdots .$$

Clearly to make the powers of ε work we need $x_0 = 1$; we then want to fix δ_1 so that

$$\exp(-[x_1\delta_1 + \cdots]) = \ln(1/\varepsilon) + \cdots .$$

For this we need

$$-[x_1\delta_1 + \cdots] = \ln\ln(1/\varepsilon) + \cdots, \qquad x_1\delta_1 = -L_2.$$

We return to the expansion:

$$x = L_1 - L_2 + x_2\delta_2 + \cdots$$

and to the governing equation:

$$\exp(-[L_1 - L_2 + x_2\delta_2 + \cdots]) = \varepsilon[L_1 - L_2 + x_2\delta_2 + \cdots]$$

$$\varepsilon L_1 \exp(-[x_2\delta_2 + \cdots]) = \varepsilon L_1 - \varepsilon L_2 + \cdots$$

$$\exp(-[x_2\delta_2 + \cdots]) = 1 - \frac{L_2}{L_1} + \cdots$$

Now since $L_2 \ll L_1$ we can assume $\delta_2 \ll 1$ and expand the exponential in the usual way:

$$1 - x_2\delta_2 + \cdots = 1 - \frac{L_2}{L_1} + \cdots$$

and so we have found

$$x_2\delta_2 = \frac{L_2}{L_1}.$$

We will carry out just one more term:

$$x = L_1 - L_2 + \frac{L_2}{L_1} + x_3\delta_3 + \cdots$$

$$\exp(-[L_1 - L_2 + L_2/L_1 + x_3\delta_3 + \cdots]) = \varepsilon[L_1 - L_2 + L_2/L_1 + x_3\delta_3 + \cdots].$$

$$x_3\delta_3 = -L_2/L_1^2 + L_2^2/2L_1^2 + \cdots .$$

In general, if logarithms appear in a problem, only trial and error (as here) or an iterative scheme (see, e.g. Hinch page 12) will give access to a solution. However, solutions are usually expressible in terms of the two logarithmic building blocks L_1 and L_2.

Warning!

Not only are logarithmic expansions horrible to find, they are also a lot less use in practice than the power series we have been looking at. Unless your physical "small parameter" is *extremely* small, L_1 will not be very large and L_2 probably not large at all: so the ordering of terms, while correct in the limit $\varepsilon \to 0$, may not be helpful at a real value of ε. The table below gives an idea of the problem.

ε	L_1	L_2	L_2/L_1	$(L_2^2 - 2L_2)/L_1^2$
10^{-1}	2.303	0.834	0.362	-0.183
10^{-3}	6.908	1.933	0.280	-0.003
10^{-5}	11.51	2.443	0.212	0.008
10^{-7}	16.12	2.780	0.172	0.008
10^{-9}	20.72	3.031	0.146	0.007