1 Dimensional Analysis

Dimensional analysis is a very simple and straighforward tool, but with a little flexibility it can be extended to create more general scaling laws: it is always worth considering at the outset of a physical problem.

1.1 Simple concepts

The fundamental units of standard dimensional analysis are the units mass, M; length, L; and time, T. There are others – but we'll start with these. So a mass has dimension M, a velocity dimension LT^{-1}, and a force dimension MLT^{-2}.

The most basic task of dimensional analysis is a check on the sanity of your equations. It is only sensible to add two quantities if they have the same dimensions. Let's look at the Navier–Stokes momentum equation as an example:

$$\rho\left(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}\right) = -\underline{\nabla}p + \eta\nabla^2\underline{u}.$$

The dimensions of the individual quantities are:

$$[\rho] = ML^{-3} \qquad [\partial_t] = T^{-1} \qquad [\underline{u}] = LT^{-1} \qquad [\nabla] = L^{-1} \qquad [p] = ML^{-1}T^{-2}$$

and let's suppose we don't know what viscosity, η, should be measured in. Then putting all these dimensions into our equation gives:

$$ML^{-3}(LT^{-2}) = ML^{-3}(LT^{-2}) = ML^{-2}T^{-2} = [\eta]L^{-2}LT^{-1}$$

and we can deduce that $[\eta] = ML^{-1}T^{-1}$.

1.2 Extending the concept

Even dimensionless numbers can be part of a dimensions system. For instance, a mole of a substance is defined to be 6.0221367×10^{23} atoms of it. Avogadro's number is just a number – it has no dimensions – and yet in scaling an equation, if one quantity is "per mole" then other quantities added to it must be too. For instance, the ideal gas law

$$pV = nRT$$

has the obvious dimensions

$$[p] = ML^{-1}T^{-2} \qquad [V] = L^3 \qquad [T] = \Theta$$

in which we are using Θ for the dimension of temperature. The other two quantities are less obvious: n is the number of moles of the substance present – just a number, but if we were to define a "dimension" for moles, $\mathcal{M}$, then we could deduce

$$[n] = \mathcal{M} \qquad [R] = ML^2T^{-2}\Theta^{-1}\mathcal{M}^{-1}$$

which does indeed fit: the value of the ideal gas constant R is

$$R = 8.314472\,\mathrm{m}^3\,\mathrm{Pa}\,\mathrm{K}^{-1}\,\mathrm{mol}^{-1} = 8.314472\,\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}\,\mathrm{mol}^{-1}.$$

This concept brings us away from pure dimensional analysis and into the realm of scaling laws.

1.3 Dimensionless parameters

The other critical technique based on dimensional analysis is nondimensionalisation. To quote Andrew Fowler,[1]

> Confronted with, or having created, a mathematical model of a continuous physical system, which consists of a set of differential equations and associated boundary condition, the first thing that an applied mathematician will want to do is non-dimensionalize the system.

Some think this desire is the only real difference between an applied mathematician and a theoretical physicist.

The principle is this: for every dimension relevant in your problem (which may include moles or other such pseudo-dimensions), pick a representative value. It may be that the natural choices are the basic dimensions M, L and T; more often they are not. For the standard example of the Navier–Stokes momentum equations above, we typically choose typical values for the three combinations

$$L \text{ lengthscale} \qquad U = LT^{-1} \text{ velocity} \qquad \eta = ML^{-1}T^{-1} \text{ viscosity.}$$

We introduce new *dimensionless variables* which are just the original variables, scaled with the relevant dimensional combinations. Thus (using a tilde ˜ to denote each dimensionless quantity) we would introduce

$$\underline{u} = U\underline{\tilde{u}} \qquad p = U\eta\tilde{p}/L$$

and scaling lengths and times gives also

$$\tilde{\partial}_t = L\partial_t/U \qquad \tilde{\nabla} = L\nabla$$

which result in the new equation (multiplying by L^2U^{-1}/η):

$$\frac{\rho UL}{\eta}\left(\frac{\partial \underline{u}}{\partial \tilde{t}} + \underline{\tilde{u}} \cdot \underline{\tilde{\nabla}\tilde{u}}\right) = -\underline{\tilde{\nabla}}\tilde{p} + \tilde{\nabla}^2\underline{\tilde{u}}.$$

We have now reduced the number of physical parameters from two (ρ and η) to just one: the Reynolds number $Re = \rho UL/\eta$. It expresses the balance between inertial and viscous terms.

Typically you will have some choice as to which variables to use for scaling and which combinations to use as your dimensionless numbers. Prior work in the field often gives the best clue here: there may be named dimensionless groups and these are often the most convenient choice. I've put a link on the website to a list of many of the fluid mechanics / heat transfer groups.

[1] Andrew C. Fowler, *Mathematical Models in the Applied Sciences*, p. 19.