Set theory and its neighbours talk, 19/11/2001.

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(Ciesielski, Larson) There is $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that for all $x \in \mathbf{R}$ there is some $d(x) > 0$ such that whenever $0 < h < d(x)$ one has $\|f(x+h) - f(x-h)\| \geq d(x)$. Such an f is *uniformly antisymmetric*. Actually, one has $f : \mathbf{R} \longrightarrow \omega$.

(Shelah) For all countable $A \subseteq \mathbf{R}$ there is some u.a. $f : A \longrightarrow \{0, 1\}$.

Question. Does there always exist such a u.a. function whose range is infinite?

For fixed f let $S_x = \{ h > 0 \mid f(x-h) = f(x+h) \}$. (So really $S_x = S_x(f)$.)

(Ciesielski-Larson) There is some $f : \mathbf{R} \longrightarrow \omega$ with S_x finite for every x.

(P.K.) There is some $f : \mathbf{Q} \longrightarrow \{0, 1, 2, 3\}$ with S_x finite for every x.

(P.K.) For every finite k, if $f : \mathbf{R} \longrightarrow k$ then some S_x is infinite.

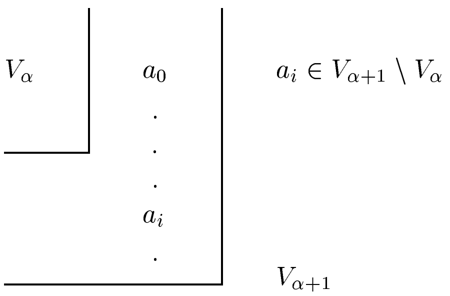
(Shelah) CH implies that there is some $f : \mathbf{R} \longrightarrow \omega$ such that $\overline{\overline{S_x}} \leq 1$ for all x.

(P.K.) If $\mathfrak{c} \geq \aleph_n$ then for every $f : \mathbf{R} \longrightarrow \omega$ there is some x with $\overline{\overline{S_x}} \geq 2^n - 1$.

(P.K.-Shelah) It is consistent (with ZFC) that $\mathfrak{c} = \aleph_n$ and there is some $f : \mathbf{R} \longrightarrow \omega$ such that $\overline{\overline{S_x}} \leq 2^n - 1$ for all $x \in \mathbf{R}$.

If CH holds then there is some $f : \mathbf{R} \longrightarrow \omega$ such that $\lim_{h \to 0} \max(f(x+h), f(x-h)) = \infty$.

Proof. Make $f(a_i) = n$ where $1/n < \|a_1 - a_0\|, \ldots, \|a_i - a_{i-1}\|$. Express $\mathbf{R}$ as a tower (of height ω_1) of countable vector subspaces (thinking of $\mathbf{R}$ as a vector space over $\mathbf{Q}$.)



If $x - h$, x, $x + h \in V_{\alpha+1}$ then whenever one of them is in $V_{\alpha+1} \setminus V_\alpha$ at least two of them are.

If $\|h\| < 1/2n$ then one of $f(x-h)$, $f(x)$ and $f(x+h)$ is $\geq n$.

(P.K.) Under MA, as well, there is some $f : \mathbf{R} \longrightarrow \omega$ such that $\lim_{h \to 0} \max(f(x+h), f(x-h)) = \infty$.

(P.K.) However it is consistent (with ZFC) that $\mathfrak{c} = \aleph_4$ and no such function exists.

Under CH there is a decomposition $\mathbf{R} = A_1 \cup A_2 \cup \ldots$ such that no 3-AP of size $1/n$ is in $A_1 \cup \ldots \cup A_n$. (A 3-AP is a three term arithmetic progression.)

(Rado) There is a decomposition $\mathbf{R} = A_1 \cup A_2 \cup \ldots$ such that no A_i contains a 3-AP.

By van der Waerden's theorem $A_1 \cup \ldots \cup A_n$ contains no $r(n)$-AP, where if $\{1, \ldots, r(n)\}$ is n-partitioned there is a 3-AP in one part. By padding the sequence of A_i with instances of the empty set one can reduce $r(n)$ to n.

Question. (Ginsburg, Linek) Can the plane be covered by 3 stars. (A *star* is a point x and a compact segment of every half line from x in the plane.

Under CH the answer is yes. But it is consistent (with ZFC) that $\mathfrak{c} = \aleph_4$ and the answer is no.

A *cloud* is a point x and finitely many points on every half-line in the place from x.

CH holds if and only if $\mathbf{R}$ is the union of 3 clouds. And if $\mathfrak{c} \leq \aleph_n$ then $\mathbf{R}$ is the union of $n + 2$ clouds.

(*cf.* Proofs of Roy Davies's generalisation of Sierpinski's decomposition theorem.)

A *circle* is a point together with *one* point on every half-line from x in the plane.

(P.K.) If a_0, a_1,... are not collinear points in $\mathbf{R}^2$ then $\mathbf{R}^2 = C_0 \cup C_1 \cup \ldots$ for some circles C_i around a_i (for $i < \omega$).

For the (negative) consistency results with $\mathfrak{c} = \aleph_4$: add ω_4 many Cohen reals. Why ω_4? Because the proofs at present use two applications of the Δ-system lemma.