Negative partition relations and pcf theory.

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1. Introduction

The starting and finishing point of this talk is the following theorem of Sierpiñski:

Theorem 1 "What Sierpiñski said." (Sierpiñski 1933).

(1) $\quad 2^{\aleph_0} \not\to [\aleph_1]^2_2$

This means: there exists a colouring f: $[2^{\aleph_0}]^2 \to \{0, 1\}$ such that $f''[H]^2 = \{0, 1\}$ for every uncountable subset H of $2^{\aleph_0}$. In other words, there is a function, or *colouring*, from the unordered pairs of real numbers into the colours {0, 1} such that every uncountable subset H of reals is panchromatic (every colour is the image of some pair from H).

The proof of (1) is very easy: it uses a *Sierpiñski partition*. Let $<_0$ be a well-ordering of the real numbers. For reals $x < y$, define $f(x, y) = 0$ if $x <_0 y$, and $f(x, y) = 1$ if $y <_0 x$. No uncountable set $H \subseteq \mathbb{R}$ can be monochromatic, since otherwise the natural ordering $<$ is a well-ordering (or an inverse well-ordering) of H, which is impossible (as $\mathbb{Q}$ is $<$-dense). ♦ ♦

From (1), it follows immediately that:

(2) $\quad \aleph_1 \not\to [\aleph_1]^2_2$.

And similar arguments also give:

(3) $\quad 2^{\kappa} \not\to [\kappa^+]^2_2$ $\quad$ for every infinite cardinal κ

This means: there exists a colouring f: $[2^{\kappa}]^2 \to \{0, 1\}$ such that $f''[H]^2 = \{0, 1\}$ for every H of size κ^+. And so too:

(4) $\quad \kappa^+ \not\to [\kappa^+]^2_2$ $\quad$ for every infinite cardinal κ.

Before continuing, let us fix the notation which will be used everywhere below and to which the reader can refer back as necessary.

Notation:
The cardinality of a set X is denoted $|X|$. Suppose α, β are cardinals. The family of α-element subsets of a set X is denoted $[X]^{\alpha} = \{Y \subseteq X : |Y| = \alpha\}$. Similarly, we define $[X]^{<\alpha} = \bigcup_{\beta<\alpha} [X]^{\beta}$.
Suppose that κ, λ are cardinals, $\aleph_0 \le \lambda \le \kappa$. The symbol $\kappa \not\to [\lambda]^{\alpha}_{\beta}$ means: there exists a function f: $[\kappa]^{\alpha} \to \beta$ such that $f''[H]^{\alpha} = \beta$ for every $H \subseteq \kappa$ of size λ.
The set of images of the α-element subsets of H under the function f is written $f''[H]^{\alpha} = \text{range}(f?[H]^{\alpha})$.
The symbol $\kappa \to [\lambda]^{<\alpha}_{\beta}$ means: if f: $[\kappa]^{<\alpha} \to \beta$, then for some $A \in [\kappa]^{\lambda}$, $f''[A]^{<\alpha} \neq \beta$.

The symbol $\kappa \not\to [\lambda]^{<\alpha}_{\beta}$ means: there exists a function f: $[\kappa]^{<\alpha} \to \beta$ such that $f''[H]^{<\alpha} = \beta$ for every $H \subseteq \kappa$ of size λ.

Far from closing a narrative, Sierpiñski's theorem (1) and its relatives (2-4) suggest many natural questions involving the optimality of the cardinals $2^{\aleph_0}$ and $\aleph_1$ and the sub and superscripts which appear. These questions, which I outline in the following paragraphs, range across contemporary set theory, touching foundational issues, and their solutions illustrate different techniques and arguments. Perhaps this richness is one reason why someone might be motivated to ask whether Sierpiñski was right, not in the banal sense whether his theorem is right or not, nor even just whether it is the best possible, but in the sense whether it captures fully the absolute core of what is going on, the "real mathematics", or the part that is impervious to forcing up to within the consistency strength of ZFC.

Since a positive partition relation of the form $\kappa \to [\kappa]^2_2$ for an uncountable cardinal κ says that κ is already a large cardinal (see Theorem 2 below), one aspect of Sierpiñski's theorem relates to the problem of proving the consistency of small cardinals (in this case at most continuum) having large cardinal properties. This theme can be developed in two ways. The first involves reflection theorems, saturated ideals and generic embeddings, as for example in various books and papers of Jech and Prikry, Magidor, Gitik and Shelah. The second avenue of exploration concerns partition properties: if P is a partition property possessed by a cardinal κ, and one adds κ generic reals, what happens to P, and to what extent does the continuum retain any traces or weak variants of P? Results about this type of question (Shelah 1996, Shelah [546]) will be mentioned at the end of the talk.

Partition properties are also related to the productivity of chain conditions. So a further aspect of Sierpiñski's theorem touches on the bounding problem for cardinal invariants: suppose that Φ is a cardinal function and $\{\mathbf{P}_i : i \in I\}$ is a family of partial orders (Boolean algebras, topological spaces, abelian groups, first-order models, etc.); what is the best bound for $\Phi(\Pi_{i\in I}\mathbf{P}_i)$ in terms of $\{\Phi(\mathbf{P}_i) : i \in I\}$? For topological spaces and Boolean algebras, this type of question is investigated in e.g. Juhasz 1980 and Monk 1996. Clearly, if a positive partition relation holds, then there is a bound, as the following example shows:

Example: If the partial orders **P** and **Q** have the κ-chain condition, and $\lambda \to [\kappa]^2_2$, then $\mathbf{P} \times \mathbf{Q}$ has the λ-chain condition.
Proof: If $\{(p_\alpha, q_\alpha) : \alpha < \lambda\}$ were an antichain in $\mathbf{P} \times \mathbf{Q}$, then a homogeneous set of size κ for the colouring f defined by $f((p_\alpha, q_\alpha), (p_\beta, q_\beta)) = 1$ iff p_α and p_β are incompatible, would contradict the κ-chain condition of the factor **P** or **Q**. ♦ ♦

Thus in ordinary set theory (ZFC), the Erdõs-Rado theorem gives the bound of $(2^{\aleph_0})^+$ for the product if P and Q have the countable chain condition (meaning the $\aleph_1$-chain condition). But whether the countable chain condition is productive (i.e. preserved under direct products) is independent of ZFC. Diamond (and even CH) implies non-productivity; Martin's Axiom implies productivity. If one reads independence phenomena as indicators of an incompletely understood underlying concept, then the search for generalisations of Martin's Axiom to higher cardinals splits into two directions. One way to generalise Martin's Axiom is

to define broader classes of forcings (Axiom A forcings, or proper forcings) and prove the consistency of appropriate internal forcing axioms, essentially answering the question what sorts of iterative operations on forcings are consistent (maybe even modulo large cardinals) [Shelah 1998, Baumgartner 1983, Mekler 1983]. This direction carries within it the potential for the inevitable reappearance of independence phenomena. However, a second direction of research is possible too, in which one replaces the question of generalising (the negation of) Martin's Axiom by the problem of proving the failure of productivity at higher cardinals. Just as we saw that positive partition relations give bounds, so the failure of productivity can be established by negative partition relations or the existence of *colourings*. In this direction, the real aim is not for relative consistency proofs (as in the first direction), but for absolute theorems of ZFC, theorems of naïve set theory. Independence phenomena indicate that one has only caught part of the real problem at hand, a relative perspective or aspect (perhaps stemming from the continuing if subliminal influence of GCH, the Oedipal trauma of Cantorian set theory), rather than an absolute intuition (Shelah 1994). Concretely, this means too that instead of the simple formulation of the bounding problem mentioned above which is sensitive to forcing deformations of the set-theoretic world associated with the GCH, one should be more interested in the absolute or pcf version (Roslanowski-Shelah 1998, Shelah-Spinas [ShSi:677], Monk 1996): given a cardinal invariant Ψ and a class $\{\mathbf{B}_i : i \in I\}$ (of Boolean algebras, partial orders, topological spaces, models of a first-order theory, …), how are $\{\Psi(\Pi \mathbf{B}_i/D) : D \text{ is an ultrafilter on } I\}$ and $\{\Pi \Psi(\mathbf{B}_i)/D : D \text{ is an ultrafilter on } I\}$ related?

If one raises the sub and/or superscripts and considers the negative partition relations $\kappa \not\to [\kappa]^{\omega}_{\kappa}$ and $\kappa \not\to [\kappa]^{<\omega}_{\kappa}$, then there are outright classical theorems of ZFC.

The talk covers some of these ideas and is in three sections, organised around three cases of negative partition relations:

$$\kappa \not\to [\kappa]^{\omega}_{\kappa};$$

$$\kappa \not\to [\kappa]^{<\omega}_{\kappa};$$

$$\kappa \not\to [\lambda]^{2}_{\beta}$$

where $\kappa, \lambda \in \{\aleph_1, \aleph_{\omega+1}, 2^{\aleph_0}\}$ and $\beta \in \omega \cup \{\aleph_0, \aleph_1, \aleph_{\omega+1}\}$. In fact, the results stated for $\aleph_{\omega+1}$ are valid for many other cardinals. For simplicity, I have stuck to one paradigmatic example. When it is possible, a short proof is given (usually taken from the exemplary expository works of others, especially Burke-Magidor 1990 and Erdõs-Hajnal-Maté-Rado 1984). Some of the characteristic methods of proof below are described:

- Sierpiñski partitions
- lists (ZFC ± V = L ± GCH)
- stepping-up arguments
- Todorèeviæ walks
- club guessing
- possible cofinalities
- large cardinals

- forcing

Some of these techniques have already been discussed by Mirna Dzamonja (club guessing) and James Cummings at other sessions of *Set Theory and its Neighbours*, so I skip those. As mentioned above, discussion is mostly within the consistency strength of ordinary set theory (ZFC) because (at the very least) of the following fact.

Theorem 2 (Erdõs-Hajnal-Rado 1965).

If $\kappa \rightarrow [\kappa]^2_2$, then κ is a (strongly) inaccessible cardinal.

Proof: If $\lambda^+ \leq \kappa \leq 2^\lambda$, then by Theorem 1, $2^\lambda \not\rightarrow [\lambda^+]^2_2$, and so $\kappa \not\rightarrow [\kappa]^2_2$. So κ is a strong limit cardinal.
If $\kappa = \cup_{i<cf(\kappa)} \kappa_i$, κ_i ? κ, $cf(\kappa) < \kappa$, then the colouring

$$f(\alpha, \beta) = 1 \text{ iff } \{\alpha, \beta\} \subseteq [\kappa_i, \kappa_{i+1}) = \{\gamma \in \text{Ord} : \kappa_i \leq \gamma < \kappa_{i+1}\}$$

can have no monochromatic set of size κ. ♦ ♦

Remark.

Using the function $f(a) = |a|$, one has:

for $\kappa \geq \lambda \geq \beta = \aleph_0$, $\kappa \not\rightarrow [\lambda]^{<\omega}_\beta$. ♦ ♦

An instance of Ramsey's Theorem $\aleph_0 \rightarrow [\aleph_0]^2_2$ is one reason we focus on the uncountable cardinals.

Section 1: If $\kappa \geq \aleph_0$, then $\kappa \not\rightarrow [\kappa]^\omega_\kappa$.

Theorem 3 (Erdõs-Hajnal 1971).

Suppose $\aleph_0 \leq \lambda \leq \kappa$. Then

(1) $\lambda \not\rightarrow [\lambda]^\lambda_{2^\lambda}$;

(2) $\kappa \not\rightarrow [\lambda]^\lambda_{2^\lambda}$.

Proof:
Part (1): List $[\lambda]^\lambda$ as $\{A_\alpha : \alpha < 2^\lambda\}$ where each A_α appears 2^λ times. For $\alpha < 2^\lambda$, choose $s_\alpha \in [A_\alpha]^\lambda \setminus \{s_\beta : \beta < \alpha\}$. Define a colouring f by:

$$f(s_\alpha) = \text{ordertype}(\{\beta < \alpha : A_\alpha = A_\beta\}, <).$$

Given $H \in [\lambda]^\lambda$ and $\xi < 2^\lambda$, let A_α be the ξ+1-st appearance of H in the list. Then $f(s_\alpha) = \xi$ and $s_\alpha \in [H]^\lambda$. ♦ ♦

Part (2): The maximal almost disjoint family argument.
For each $X \in [\kappa]^\lambda$, let $f_X : [X]^\lambda \rightarrow 2^\lambda$ be a colouring of the λ–element subsets of X as given by Part (1). Let $F \subseteq [\kappa]^\lambda$ be a maximal almost disjoint family (Zorn's Lemma), and define a colouring $g: [\kappa]^\lambda \rightarrow 2^\lambda$ by:

$$g(Z) = f_X(Z) \text{ if } (\exists X \in F)(Z \subseteq X) \text{ and } g(Z) = 0 \text{ otherwise.}$$

Check that g does the job. ♦ ♦

Corollary 4 (Erdõs-Rado 1956).

$$\kappa \not\to [\lambda]^{\lambda}_{2}.$$

Proof: Immediate.

Proposition 5 (Erdõs-Rado 1966; Kunen 1971).

If λ is a strong limit cardinal of countable cofinality, then $\lambda \not\to [\lambda]^{\omega}_{\lambda}$.

Proof: Note that $\lambda^{\aleph_0} = 2^{\lambda}$, so one can list $[\lambda]^{\aleph_0} \times \lambda$ as $\{(A_\alpha, \xi_\alpha) : \alpha < 2^{\lambda}\}$. As before, choose $s_\alpha \in [A_\alpha]^{\aleph_0} \setminus \{s_\beta : \beta < \alpha\}$ and define $f(s_\alpha) = \xi_\alpha$.
If $H \in [\lambda]^{\aleph_0}$ and $\xi < \lambda$, then there is α such that $(H, \xi) = (A_\alpha, \xi_\alpha)$, and so $f(s_\alpha) = \xi$, $s_\alpha \in [H]^{\aleph_0}$. ♦ ♦

The colourings of Proposition 5 prevent the universe from being more Borges-like than it already appears:

Corollary 6 (Kunen 1971).
There is no non-trivial elementary embedding $j: V \to V$ of the universe into itself.

Proof: Otherwise, let κ be the first ordinal moved by $j: V \to V$, and define $j^1 = j$, $j^{n+1} = j \circ j^n$. Let $\lambda = \sup\{j^n(\kappa) : n < \omega\}$. Note that $j(\lambda) = \lambda$ and $j(\omega) = \omega$. Also $j^n(\kappa)$ is measurable, so λ is a strong limit cardinal of cofinality ω. Using the colouring f from Proposition 5, elementarity, the set $H = \{j(\alpha) : \alpha < \lambda\}$ and $\kappa < \lambda$, there exists $s \in [H]^{\omega}$, $j(f)(s) = \kappa$. So there exists $t \in [\lambda]^{\omega}$ with $j(t) = s$. But now: $\kappa = j(f)(s) = j(f)j(t) = j(f(t)) \geq j(\kappa) > \kappa$. Contradiction. ♦ ♦

Definition:

A *set mapping* over a set X is a function $f: \mathrm{dom}(f) \subseteq P(X) \to P(X)$ such that $Y \cap f(Y) = \varnothing$ for every $Y \in \mathrm{dom}(f)$.

A subset H of X is *free with respect to f* if $H \cap f(Y) = \varnothing$ for every $Y \in P(H) \cap \mathrm{dom}(f)$.

The symbol $(\kappa, \lambda, \alpha) \to \beta$ means: for an arbitrary set mapping $f: [\kappa]^{\lambda} \to [\kappa]^{<\alpha}$, there exists a free subset H of size β.

Theorem 7 (Erdõs-Hajnal 1958).

For all κ and λ,

(1) $(\kappa, \lambda, 2) \not\to \lambda$.

(2) $(\kappa, \aleph_0, 2) \not\to \aleph_0$.

Proof: Rather like $\kappa \not\to [\lambda]^{\lambda}_{2^{\lambda}}$; first one proves $(\lambda, \lambda, 2) \not\to \lambda$. Let h: $[\lambda]^{\lambda} \to [\lambda]^{\lambda}$ be an injective mapping such that $h(X) \subset X$ (proper inclusion) for all $X \in [\lambda]^{\lambda}$. Put $f(Y) = \{g(Y)\}$ for $Y \in [\lambda]^{\lambda}$. No $H \in [\lambda]^{\lambda}$ is free with respect to f [why? $G(h(H)) \in H \setminus h(H)$, so $h(H) \in P(H) \cap \mathrm{dom}(f)$, and $H \cap f(h(H)) = H \cap \{g(h(H))\} \neq \emptyset$.]

To prove $(\kappa, \lambda, 2) \not\to \lambda$, use a maximal almost disjoint family argument. ♦ ♦

Theorem 8 (Erdõs-Hajnal 1966).

In every infinite cardinal κ, there exists an infinitary Jonsson algebra with finitely many operations, i.e.
there exists an algebra $A = (A, F)$ such that $|A| = \kappa$, F is a finite set of operations on A which may have ω-many arguments, and A has no proper subalgebra of size κ.

Proof: By Theorem 7, there is a set mapping $g: [\kappa]^{\aleph_0} \to [\kappa]^1$ which has no infinite free subset. Let $f: [\kappa]^{\aleph_0} \to \kappa$ be given by $\{f(x)\} = g(x)$.

Suppose (for a contradiction) that the conclusion of Theorem 8 is false. We construct an infinite free subset as follows.
Define, by induction, distinct ordinals $\langle \xi_k : k < \omega \rangle$, and $\langle A_k : k < \omega \rangle$ such that:
$|A_k| = \kappa$, $A_0 = \kappa$, $A_{k+1} \subseteq A_k$, $\{\xi_i : i < k\} \cap A_k = \emptyset$.
How? Given $\{\xi_i : i < k\} \cap A_k = \emptyset$, for $b \subseteq \{\xi_i : i < k\}$ and $x \in [\kappa \setminus b]^{\omega}$, put $f_b(x) = f(b \cup x)$. Since the algebra $\mathbf{A}_k = (A_k, f_b)_{b \subseteq \{\xi_i : i < k\}}$ is not Jonsson, $\exists \xi_k \in A_k$ and $A_{k+1} \subseteq A_k \setminus \{\xi_k\}$, $|A_{k+1}| = \kappa$ such that $\forall b \subseteq \{\xi_i : i < k\}$ $\forall x \in [A_{k+1}]^{\omega}$, $\xi_k \neq f_b(x)$. Now $H = \{\xi_k : k < \omega\}$ is free with respect to g – contradiction. ♦ ♦

Theorem 9 (Erdõs-Hajnal 1966).

For every infinite κ, $\kappa \not\to [\kappa]^{\omega}_{\kappa}$.

Proof: By Theorem 8, there is an infinitary Jonsson algebra $(\kappa, f_0, \ldots, f_n)$, where f_i may be ω-ary. Code the operations $f_0, \ldots, f_n$ as a single operation $f: [\kappa]^{\omega} \to \kappa$. The function f is the required colouring. ♦ ♦

I think that there are still some questions like (Erdõs-Hajnal-Maté-Rado 1984), although they are slightly removed from the focus of this talk:

QUESTION: Does GCH imply $\aleph_{\omega+1} \not\to [\aleph_2]^{\omega}_{\omega_2}$.

QUESTION: Without GCH, does $\lambda \not\to [(2^{\aleph_0})^+]^{\omega}_{(2^{\omega})^+}$ hold for any $\lambda > (2^{\aleph_0})^+$?

Definition:

A cardinal κ is *super-compact* if for all ordinals α, there is a transitive model M and an elementary embedding j: $V \to M$ such that:
(1) ${}^{\alpha}M \subseteq M$;
(2) $j(\xi) = \xi$ for all $\xi < \kappa$;
(3) $j(\kappa) > \kappa$.

The combinatorial property *P(λ)* is the assertion: whenever f: $[\lambda]^{\omega} \to \lambda$, there exists an infinite subset $A \subseteq \lambda$ such that $|A|$? $f''[A]^{\omega}$.

Theorem 10 (Kunen 1971).

If κ is super-compact, then $P(\kappa^+)$ holds.

Section 2: $\kappa \not\to [\kappa]^{<\omega}_{\kappa}$.

Definition.

An algebra $A = (A, f_n)_{n \in \omega}$ is a *Jonsson algebra* if *A* has no proper subalgebra of size $|A|$.

Example 11.

$(\omega, f(x) = x - 1)$ is a Jonsson algebra on ω. ♦ ♦

Fact 12 (Shelah 1994).

For a cardinal λ, TFAE:

(1) there is a Jonsson algebra on λ;
(2) $\lambda \not\to [\lambda]^{<\omega}_{\lambda}$;
(3) $\lambda \not\to [\lambda]^{<\omega}_{\sigma}$ for every $\sigma < \lambda$;
(4) if $\chi > \lambda$, M ? $H(\chi)$, $\lambda \in M$, $|M \cap \lambda| = \lambda$, $\sigma \in M$ and $\sigma^+ < \lambda$, then $\sigma \subseteq M$;
(5) for some (equivalently for all) regular $\theta \geq \lambda^+$ and for all M ? $H(\theta)$, one has:
if $\lambda \in M$ and $|M \cap \lambda| = \lambda$, then $\lambda \subseteq M$.

Proof: The equivalences that we shall use are (1), (2) and (5).

(1) $\Leftrightarrow$ (2): Use the colouring. Add Skolem functions and close under composition.

(1) $\Rightarrow$ (5): Since Jonsson algebras on λ belong to $H(\theta)$ (the family of sets of cardinality hereditarily less than θ), by elementarity there is on λ a Jonsson algebra $\mathbf{A} = (A, f_i)_{i<\omega} \in M$.

Let B = M ∩ λ. Note |B| = |M ∩ λ| = λ, and **B** = (B, $f_i \lceil B)_{i<\omega}$ is a subalgebra of **A**. Hence B = A = λ, i.e. λ ∩ M = λ, so λ ⊆ M.

(5) ⇒ (1): Fix an elementary submodel M ? H(θ), |M| = λ, λ ⊆ M. Let h: λ → M be a bijection. Consider $\mathbf{M}^+$ = (M, ∈, h).
Claim: $\mathbf{M}^+$ is a Jonsson model.
If **N** ? $\mathbf{M}^+$ and |N| = λ, then λ ∈ N (because $\mathbf{M}^+ ¦ \lambda = \min\{\alpha : \alpha \notin \text{dom}(h)\}$ and **N** ? $\mathbf{M}^+$).
Also |N ∩ λ| = λ since |N ∩ λ| = $|h^{-1}[N]|$ = |N|. So by hypothesis, λ ⊆ N. Hence range(h) ⊆ N.
But M = range(h). So **N** = $\mathbf{M}^+$. Add Skolem functions to make the model into an algebra. ♦ ♦

Theorem 13 (Negative Stepping Up).

If there is a Joñsson algebra on λ, then there is a Joñsson algebra on λ^+.

Proof: We use Fact 12 twice. Suppose **M** is an elementary submodel of $H(\lambda^{++})$, $\lambda^+ \in M$, and $|M \cap \lambda^+| = \lambda^+$. We show $\lambda^+ \subseteq M$.

Note that λ ∈ M (since $\lambda^+ \in M$), and $(\exists \alpha \in M \cap \lambda^+)(|M \cap \alpha| = \lambda)$. Now M contains a bijection g: λ → α. Hence |M ∩ λ| = λ. There is a Jonsson algebra on λ, so (by Fact 12), λ ⊆ M. Therefore α = range(g) ⊆ M. It follows that $\lambda^+ \subseteq M$.

By Fact 12 again, there is a Jonsson algebra on λ^+. ♦ ♦

Corollary 14.

For every natural number n, there is a Joñsson algebra on $\aleph_n$.

Theorem 15 (Tryba 1984; Woodin).

If λ is a regular cardinal and there is a non-reflecting stationary subset S of λ, then there is a Joñsson algebra on λ.

In particular, there is a Joñsson algebra on κ^+ whenever κ is regular.

Proof: Use Fact 12. Suppose **M** is an elementary submodel of $H(\lambda^+)$, λ ∈ M, and |M ∩ λ| = λ. We show λ ⊆ M.

Let E ∈ M be a non-reflecting stationary subset and let C = {α < λ : sup(M ∩ α) = α}. Note that C is club in λ. Claim: C ∩ E ⊆ M. [Why? Otherwise, ∃α ∈ (C ∩ E) \ M. Let γ = min{β ∈ M : β > α}. Note γ = ∪γ. There exists a club C_γ of γ such that C_γ ∩ E = ∅. ∀β ∈ M ∩ α, [β, γ) ∩ C_γ ∩ M ≠ ∅. So C_γ ∩ α is unbounded in α, and hence α ∈ C_γ, so α ∈ C_γ ∩ E - a contradiction.] Now write the non-reflecting stationary set S as a disjoint union of stationary sets, S = $\cup_{\alpha<\lambda} S_\alpha$. By the previous Claim, C ∩ S_α ⊆ M, so C ∩ S_α ≠ ∅ and now this implies that α ∈ M for all α < λ. So λ ⊆ M. ♦ ♦

QUESTION: Can one prove in ZFC that there is a Jonsson algebra on every successor cardinal?

Theorem 16 (Keisler, Rowbottom).

The axiom of constructibility V = L implies that there is a Jońsson algebra on every infinite cardinal.

Proof: Apparently sharps mean different things to different people. Basically, the Jońsson algebra on κ is the model $(L_\kappa, \in)$ with Skolem functions added. ♦ ♦

Corollary 14 leaves open the question what happens at $\aleph_{\omega+1}$ (for example) which is dealt with next.

Definition.

Suppose that *a* is a set of regular cardinals. The set of *possible cofinalities* of *a*, denoted pcf(*a*), is defined as follows:

$$\mathrm{pcf}(a) = \{\mathrm{tcf}(\textstyle\prod a/D) : D \text{ is an ultrafilter on } a\}.$$

The *true cofinality* of $\prod a/D$ is the least λ such that there exists a $<_D$-increasing sequence $\langle f_\alpha \in \prod a : \alpha < \lambda\rangle$ such that for every $g \in \prod a$, there exists $\alpha < \lambda$ such that $g \leq_D f_\alpha$.

Generally assume that $|a| < \min(a)$ and *a* is a set of regular cardinals.

QUESTION (Shelah 1994): Is |pcf(a)| = |a| ?

Example.
Let $a = \{\aleph_n : 1 < n < \omega\}$. Then $a \subseteq \mathrm{pcf}(a)$ (using principal ultrafilters). Also, if D is a non-principal ultrafilter, then $\mathrm{tcf}(\prod a/D) \geq \aleph_{\omega+1}$. ♦ ♦

Theorem 17 (Shelah 1994).

$$\aleph_{\omega+1} \in \mathrm{pcf}(\{\aleph_n : 1 < n < \omega\}).$$

In a wildly loose way, one might interpret Theorem 17 as a sort of prediction principle: it says that some ultrafilter D and cofinal guessing sequence $\langle f_\alpha/D : \alpha < \omega_{\omega+1}\rangle$ work to predict, for any $f \in \Pi\, \aleph_n$, bounds (on a large set $X \in D$) defined by some f_α.

Theorem 18 (Shelah 1994).

There is a Jońsson algebra on $\aleph_{\omega+1}$.

Proof: Let $\mu = \aleph_\omega$, $\theta = (2^\mu)^+$, and fix an elementary submodel **M** of $H(\theta)$ with $\mu^+ \in M$ and $|M \cap \mu^+| = \mu^+$. By Fact 12, it is enough to show that $\mu^+ \subseteq M$. By Theorem 17, $\mu^+ \in \mathrm{pcf}(a)$ where $a = \{\aleph_n : 1 < n < \omega\}$. By elementarity, one has:

(1) $a \in M$
(2) $a \subseteq M$
(3) there is an ultrafilter D on a, $D \in M$, there exists $\langle f_\beta : \beta < \mu^+\rangle \in M \cap \prod a$ such that $\langle f_\beta/D : \beta < \mu^+\rangle$ is increasing and cofinal in $\prod a/D$.

Claim: $\{\alpha \in a : |M \cap \alpha| = \alpha\}$ is cofinal in a. [Why? Otherwise $g(\alpha) = \sup(M \cap \alpha)$ (= 0 if $\sup(M \cap \alpha) = \alpha$) defines an element in $\prod a$, and now by (3) there is β with $g/D < f_\beta/D$. Since D is non-principal, there is α such that $0 < g(\alpha) < f_\beta(\alpha)$. But $f_\beta(\alpha) \in M \cap \alpha$ - a contradiction.) By Corollary 14, there is a Jońsson algebra on α for each $\alpha \in a$. So by (2), $\alpha \subseteq M$ (Fact 12 again). Thus $\aleph_\omega = \mu \subseteq M$.
Finally, for $\xi \in M \cap [\mu, \mu^+)$, there is a bijection $f \in M$ from μ onto ξ, and hence $\xi \subseteq M$. But $|M \cap \mu^+| = \mu^+$, hence $\mu^+ \subseteq M$. So there is a Jońsson algebra on $\aleph_{\omega+1}$. ♦ ♦

Theorem 19 (Shelah 1992).

The Strong Hypothesis $pp(\lambda) = \lambda^+$ for singular cardinals λ implies that there exists a Jońsson algebra in every successor cardinal.

Section 3: Wasn't Sierpiński right?

Finally let us return to Sierpinski's theorem (Theorem 1), now that the situation for the sub and superscripts has been explored somewhat.

Theorem 20 (Erdős-Hajnal-Milner 1968).

(1) The Continuum Hypothesis implies
$$\aleph_1 \not\to [\aleph_1]^2_{\omega_1}.$$

(2) The Generalised Continuum Hypothesis implies: there is a function $f: [\kappa]^2 \to \kappa^+$ such that for every $X \in [\kappa^+]^\kappa$ for every $Y \in [\kappa^+]^{\kappa^+}$ there exists $\xi \in X$ with $\{f(\xi, \alpha) : \alpha \in Y\} = \kappa^+$.

(3) GCH implies $\kappa^+ \not\to [\kappa^+]^2_{\hat{e}^+}$.

Proof of (2): Using $2^\kappa = \kappa^+$, list the functions h: $dom(h) \in [\kappa^+]^\kappa \to \kappa^+$ as $\{h_\alpha : \alpha < \kappa^+\}$.
For $\alpha < \kappa^+$, let $g(\alpha, -): \rho_\alpha \to \alpha$ be a surjection from some cardinal ρ_α to the set $\{\beta < \alpha : dom(h_\beta) \subseteq \alpha\}$. Note $\rho_\alpha \leq \kappa$.
For $\nu < \rho_\alpha$, pick $\xi_{g(\alpha, \nu)}$ to be an element of $dom(h_{g(\alpha, \nu)}) \setminus \{\xi_{g(\alpha, \mu)} : \mu < \nu\}$. This is possible since $\nu < \rho_\alpha \leq \kappa$ and $|dom(h_{g(\alpha, \nu)}| = \kappa$.

NOW, define a colouring $f: [\kappa^+]^2 \to \kappa^+$ as follows: if $\alpha < \kappa^+$, $\beta < \alpha$, $dom(h_\beta) \subseteq \alpha$, and $\beta = g(\alpha, \nu)$ for some $\nu < \rho_\alpha$, put $f(\xi_\beta, \alpha) = h_\beta(\xi_\beta)$; otherwise f has the value 0.

Claim: f works.
Why? Given $X \in [\kappa^+]^\kappa$, $Y \in [\kappa^+]^{\kappa^+}$, suppose that:

(*) $(\forall \xi \in X)(\exists h(\xi) \neq f(\xi, \alpha)$ for all $\alpha \in Y)$.

Then $dom(h) = X \in [\kappa^+]^\kappa$ and $h = h_\beta$ for some $\beta < \kappa^+$.

There exists $\alpha < \kappa^+$, $\beta < \alpha$, $\alpha \in Y$, $X \subseteq \alpha$ (because $|Y| = \kappa^+$). So $\mathrm{dom}(h_\beta) \subseteq \alpha$ and $\beta = g(\alpha, \nu)$ for some $\nu < \rho_\alpha$. It follows that: $f(\xi_\beta, \alpha) = h_\beta(\xi_\beta) = h(\xi_\beta)$, and $\xi_\beta = \xi_{\gamma(\alpha, \nu)} \in \mathrm{dom}(h_\beta) \subseteq X$. Thus $\xi_\beta \in X$, $\alpha \in Y$, and $f(\xi_\beta, \alpha) = h(\xi_\beta)$ – contradiction to (*). ♦ ♦

Parts (1) and (3) are immediate from (2). ♦ ♦

QUESTION: Can one prove 20.2 without GCH?

Corollary 21 (Erdõs-Hajnal 1965).

The Generalised Continuum Hypothesis implies that there is a Joñsson algebra on λ^+ for every cardinal λ.

Proof: This is evident from Theorem 20 (2): just use (κ^+, f) where f is the colouring above. ♦ ♦

"Problem 15" (Erdõs-Hajnal 1971):

Can one prove without CH any of the following:

$2^{\aleph_0} \not\to [\aleph_1]^2_3$ or $2^{\aleph_0} \not\to [2^{\aleph_0}]^2_2$ or $\aleph_1 \not\to [\aleph_1]^2_3$.

Theorem 22 (Galvin-Shelah 1973).

(1) $\aleph_1 \not\to [\aleph_1]^2_4$;

(2) $2^{\aleph_0} \not\to [2^{\aleph_0}]^2_{\aleph_0}$.

We shall give just the proof of part two of Theorem 22, since substantially stronger results (Theorems 24 and 25) make the first part of Theorem 22 obsolete (although the proofs are different). The proof of Theorem 22 proceeds through a sequence of lemmas.

Lemma 22.1.

Suppose that $f_0, \ldots, f_n$ are functions from $2^{\aleph_0}$ to $\mathbb{R}$, and for every cofinal subset B of $2^{\aleph_0}$, there exist $\mu, \nu \in B$ such that $\forall\, i \le n$, $f_i(\mu) < f_i(\nu)$. Then there exists an injective function $g: 2^{\aleph_0} \to \mathbb{R}$ such that:

whenever B is cofinal in $2^{\aleph_0}$, then

(1) $\exists\, \mu, \nu \in B$, $\forall\, i \le n$, $f_i(\mu) < f_i(\nu)$ and $g(\mu) > g(\nu)$;

(2) $\exists\, \mu, \nu \in B$, $\forall\, i \le n$, $f_i(\mu) < f_i(\nu)$ and $g(\mu) < g(\nu)$.

Proof: Let $\{A_\alpha : \alpha < 2^{\aleph_0}\}$ list the countably infinite subsets of $2^{\aleph_0}$. Define $g: 2^{\aleph_0} \to \mathbb{R}$ by induction in the following way.

Given $\{g(\alpha) : \alpha < \beta\}$, if $A_\alpha \in [\beta]^\omega$, let $x(\alpha) = \sup\{g(\xi) : \xi \in A_\alpha, \forall i \leq n, f_i(\xi) < f_i(\beta)\}$, and $y(\alpha) = \inf\{g(\xi) : \xi \in A_\alpha, \forall i \leq n, f_i(\xi) < f_i(\beta)\}$.
Choose now $g(\beta) \in \mathbb{R} \setminus (g(\alpha), x(\alpha), y(\alpha) : \alpha < \beta\}$.
Some effort will show that the function g works. ♦ ♦

Lemma 22.2.

There are injective functions f_i: $2^{\aleph_0} \to \mathbb{R}$ such that for every cofinal subset B in $2^{\aleph_0}$ and $n < \omega$, $\exists \mu, \nu \in B, \forall i \leq n, f_i(\mu) < f_i(\nu)$ and $f_{n+1}(\mu) > f_{n+1}(\nu)$.

Proof: Iteratively apply Lemma 22.1 to some injective function f_0: $2^{\aleph_0} \to \mathbb{R}$ also given by Lemma 22.1. ♦ ♦

Proof of Theorem 22: Define a colouring f: $[2^{\aleph_0}]^2 \to \omega$ by:
$f(\alpha, \beta) = n$ iff $\forall i \leq n, f_i(\alpha) < f_i(\beta)$ and $f_{n+1}(\alpha) > f_{n+1}(\beta)$, where $\{f_i : i < \omega\}$ is given by Lemma 22.2. Clearly, f does the trick. ♦ ♦

Limits on Theorem 22 part (2) are given by:

Theorem 23 (Solovay 1971).

Suppose that $2^{\aleph_0}$ is real-valued measurable.

(1) $2^{\aleph_0} \to [2^{\aleph_0}]^2_{\aleph_1}$

(2) for all $n < \omega$, $2^{\aleph_0} \to [2^{\aleph_0}]^n_{\aleph_1}$.

Proof sketch: Use the σ-completeness of the ideal to lift the proof of Ramsey's theorem. ♦ ♦

The real-valued measurable continuum is large, so there is still hope for colourings on small successor cardinals:

Theorem 24 (Todorèeviæ 1987).

$$\aleph_1 \not\to [\aleph_1]^2_{\omega_1}.$$

We sketch the proof of a more general result:

Theorem 25 (Todorèeviæ 1987; Shelah 1988).

Suppose λ is an uncountable regular cardinal and there is a non-reflecting stationary subset S of λ, then $\lambda \not\to [\lambda]^2_\lambda$.

Sketch of Sketch of Proof: The notation means that we want to define a colouring d: $[\lambda] \to \lambda$ such that $\forall H \in [\lambda]^\lambda \forall \xi < \lambda \exists \alpha, \beta \in H, \alpha < \beta$, and $d(\alpha, \beta) = \xi$. The definition of d uses (simplified) *Todorèeviæ walks*. For each ordinal $\iota < \lambda$, choose $C(\iota)$ such that for non-zero $\iota < \lambda$, if ι is a successor, then $C(\iota) = \{\iota - 1, 0\}$, and if ι is a limit ordinal, then $C(\iota)$ is a club of ι

disjoint from S (possible because S is non-reflecting), $0 \in C(\iota)$, and successors in $C(\iota)$ are successors in λ. So, if $\delta \in S$, $0 < \iota < \lambda$, then $\delta \in C(\iota)$ iff $\iota = \delta + 1$.
By Solovay's theorem, one can write S as a partition into stationary sets $S = \cup_{\xi < \lambda} S_\xi$.

Fix $\alpha < \beta < \lambda$. The walks $\gamma^+_n(\beta, \alpha)$ and $\gamma^-_n(\beta, \alpha)$ from β to α are constructed recursively as follows:

$\gamma^+_0(\beta, \alpha) = \beta$, $\gamma^-_0(\beta, \alpha) = 0$;

if $\gamma^+_l(\beta, \alpha) \neq \infty$ and is greater than α, then $\gamma^+_{l+1}(\beta, \alpha) = \min C(\gamma^+_l(\beta, \alpha)) \cap [\alpha, \lambda)$, and $\gamma^-_{l+1}(\beta, \alpha) = \sup \mathrm{cl}(C(\gamma^+_{l+1}(\beta, \alpha) \cap \alpha)$ (i.e. the closure in the order topology).

Let $k = k(\beta, \alpha) = \min\{l : \gamma^+_l(\beta, \alpha) = \alpha\}$.

For $m \leq k(\beta, \alpha)$, let $\varepsilon_m(\beta, \alpha) = \max\{\gamma^-_l(\beta, \alpha) + 1 : l \leq m\}$. So $\varepsilon_m(\beta, \alpha) \leq \alpha + 1$.

For $\varepsilon_m(\beta, \alpha) \leq \xi \leq \alpha$, $l \leq m$, one has $\gamma^+_l(\beta, \alpha) = \gamma^+_l(\beta, \xi)$ and $\gamma^-_l(\beta, \alpha) = \gamma^-_l(\beta, \xi)$.

Let $m \leq k(\beta, \alpha)$ be maximal such that:
(1) $\varepsilon_m(\beta, \alpha) < \alpha$;
(2) $\gamma^-_l(\alpha, \varepsilon_m(\beta, \alpha)) = \gamma^-_l(\beta, \varepsilon_m(\beta, \alpha))$ for all $l \leq m$;
(3) $\gamma^+_m(\beta, \alpha) \in S$.

NOW define the colouring $d(\beta, \alpha) = \xi$ iff $\gamma^+_m(\beta, \alpha) \in S_\xi$ (0 otherwise). It remains to check that d is as required.

Given fixed $H \in [\lambda]^\lambda$ and $\xi < \lambda$, consider the model $\mathbf{M} = (\lambda, \in, H, C)$,
where $(\delta, \iota) \in C$ iff $\delta \in C(\iota)$ (viewing C as a binary predicate, H as a unary predicate).

Let $\langle N_\iota : \iota < \lambda \rangle$ be an increasing chain of elementary submodels of M, $|N_\iota| < \lambda$, $\iota \in N_{\iota+1}$. The set $A = \{\delta < \lambda : \xi < \delta$ and $\|N_\delta\| = \delta = \cup\delta\}$ is a club of λ, so there exists $\delta \in S_\xi \cap A$.
Choose $\beta \in H$, $\beta > \sup N_{\delta+1}$. Now $\delta < \beta$ implies $k(\beta, \delta) \neq \infty$ and $k(\beta, \delta) > 0$. Let $\varepsilon = \varepsilon_{k(\beta, \delta)}$.
We know that $\gamma^-_{k(\beta,\delta)}(\beta, \delta) = \delta$ iff δ is an accumulation point of $C(\gamma^+_{k(\beta,\delta)}(\beta, \delta))$. It's easy enough to show that $\varepsilon < \delta$, so ε can be a parameter in N_δ. Let $\varphi(x, y, \varepsilon, \gamma^-_l(\beta, \delta))$ be the formula saying "x is a limit ordinal and $y \in H$ and $\varepsilon < x < y$ and $\gamma^-_l(y, x) = \gamma^-_l(\beta, \delta)$".
Let $\psi(x, y)$ be the formula $(\forall\, l \leq k(\beta, \delta))[\varphi(x, y, \varepsilon, \gamma^-_l(\beta, \delta)) \wedge \gamma^+_{k(\beta,\delta)}(y, x) = x]$.
Then $\mathbf{M} \upharpoonright \psi(\beta, \delta)$, and (*) $\mathbf{M} \upharpoonright \forall\, x\, \exists\, x'\, (x' > x \wedge \forall\, y\, \exists\, y'\, (y' > y)\, \psi(y', x'))$.
So (**) $N_\delta \upharpoonright \forall\, x\, \exists\, x'\, \forall\, y\, \exists\, y'\, \psi(y', x')$. Now take ξ for x and use (**) to get $\delta' < \delta$ such that (***) $N_\delta \upharpoonright \delta' > \xi \wedge \forall\, y\, \exists\, y'\, \psi(y', \delta')$. Find an element of $C(\delta) \cap [\delta', \lambda)$ and $\beta' \in N_\delta$, $\beta' > \delta'$, such that $N_\delta \upharpoonright \psi(\beta', \delta')$ and $C(\delta) \cap [\delta', \beta') \neq \varnothing$.

Final claim: $d(\beta, \beta') = \xi$. Just check that $m = k(\beta, \delta)$ is the maximal ordinal less than or equal to $k(\beta, \beta')$ such that:

(1) $\varepsilon* = \varepsilon_m(\beta, \beta') < \beta'$;
(2) $\gamma_l^-(\beta', \varepsilon*) = \gamma_l^-(\beta, \varepsilon*)$ for all $l \leq m$;
(3) $\gamma_m^+(\beta, \beta') \in S_\xi$.

This completes the proof. ♦ ♦

However, Theorems 24 and 25 do not resolve the case of successor of singular. The simplest case is whether $\aleph_{\omega+1} \not\to [\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}$.

Theorem 26 (Todorèeviæ 1987).

$$\aleph_{\omega+1} \not\to [\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}.$$

Proof: Let $a = \{\aleph_n : 1 < n < \omega\}$, $\lambda = \sup(a)$. By Theorem 17, $\lambda^+ \in pcf(a)$, so there is a non-principal ultrafilter D on the set a and a cofinal $<_D$-increasing sequence $\{f_\alpha : \alpha < \lambda^+\}$. By Theorem 25, for each $\aleph_n \in a$, $\aleph_n \not\to [\aleph_n]^2_n$, so there exists a colouring c_n witnessing this fact. Define c: $[\lambda^+]^2 \to \lambda^+$ by: $c(\alpha, \beta) = c_n(f_\alpha(n), f_\beta(n))$, where $n = \min\{m : f_\alpha(m) \neq f_\beta(m)\}$. Given $H \in [\lambda^+]^{\lambda^+}$ and $\xi < \lambda$, note that $(\exists n)(\xi < \aleph_n$ and $|\{f_\alpha(n) : \alpha \in H\}| = \aleph_n)$.
Also $(\exists\, s \in \Pi_{i<n} \aleph_i)(|\{f_\alpha(n) : f_\alpha ?n = s, \alpha \in H\}| = \aleph_n)$. So there are $\alpha, \beta \in H$, $\alpha < \beta$, such that $c_n(f_\alpha(n), f_\beta(n)) = \xi$. Thus $c(\alpha, \beta) = \xi$. We still need to manipulate the range of c a little. Let e_β: $\beta \to \lambda$ be an injection (for each $\beta < \lambda^+$), and define

$$c^+(\alpha, \beta) = e_\beta^{-1}(c(\alpha, \beta))$$

when $\alpha < \beta$ and $c(\alpha, \beta) \in$ range(e_β). Now c^+ witnesses $\aleph_{\omega+1} \not\to [\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}$. ♦ ♦

Theorem 27 (Shelah 1988; 1996).

It is consistent that $2^{\aleph_0} \to [\aleph_1]^2_3$ (assuming some large cardinals).

This proof is not so easy to explain in five lines, but very vaguely and maybe inaccurately stated, the idea is to start from a model of ZFC in which a cardinal κ has a positive partition property (e.g. κ is a measurable or an Erdõs or strongly inaccessible Mahlo cardinal, depending on when you tuned in to the programme); do some preliminary forcing to make some desirable combinatorial property of κ indestructible relative to some appropriate class of forcings; add κ generic reals using a suitable partial order.

New questions arise: what about

(1) $\aleph_2 \to [\aleph_1]^2_3$
(2) $2^{\aleph_0} \to [\aleph_1]^3_{\aleph_0}$
(3) $2^{\aleph_0} \to [\aleph_2]^2_3$

(4) $\lambda \rightarrow [\lambda]^2_{\aleph_0}$, if λ is not weakly compact.

There is (= was) also a conjecture of Galvin:

For a suitable function h: $\omega \to \omega$, it is consistent that

$$2^{\aleph_0} \geq \aleph_2 \text{ and } \aleph_2 \to [\aleph_1]^n_{h(n)} \text{ for all } n < \omega.$$

Theorem 28 (Shelah 1991).

(1) The Galvin Conjecture is false.

(2) For every $n < \omega$, for some $m < \omega$,
$$\aleph_n \to [\aleph_1]^m_{\aleph_0}.$$

(3) It is consistent $2^{\aleph_0} \to [\aleph_1]^n_{h(n)}$ holds for a suitable function h for every $n < \omega$, assuming some large cardinals.

Strong negative partitions around the continuum.

Problem: what is "the real hard-core generalization" of Martin's Axiom *MA*?

There is a discussion of this problem and its history in e.g. Shelah 1994.

Theorem (Martin-Solovay).

Martin's Axiom implies that the c.c.c. (countable chain condition) is preserved under products.

Definition.

If λ is a cardinal, the symbol $Pr^+(\lambda)$ means that there is a function $c: [\lambda]^2 \to \lambda$ such that if $1 \leq n < \omega$ and the sets $\{ \varsigma^0_\alpha, ..., \varsigma^{n-1}_\alpha \}$ are disjoint for $\alpha < \lambda$ and $\varsigma^0_\alpha < ... < \varsigma^{n-1}_\alpha$, then for every $h: n \times n \to \lambda$, there are $\alpha < \beta$ such that $c(\varsigma^i_\alpha, \varsigma^j_\beta) = h(i, j)$ for $i, j < n$.

The symbol $Pr(\lambda)$ has a similar meaning but only for every constant function $h: n \times n \to \lambda$, i.e. $h(i, j) = \gamma$ for some $\gamma < \lambda$.

Note that $Pr(\lambda)$ implies $\lambda \not\to [\lambda]^2_\lambda$ for regular λ.

Theorem (Shelah 1991; 1994).

Suppose there is a non-reflecting stationary subset S of λ such that $cf(\alpha) > \omega_1$ for every $\alpha \in S$. Then $Pr(\lambda)$ holds.

Even more can be shown.

Corollary (Shelah 1990; 1991; 1994).

Strong negative partition relations $Pr_l(\lambda, \lambda, \kappa, \theta)$, $l = 0, 1, \ldots, 3$, $\kappa \geq 2$, $\theta \geq 2$ imply:

(1) $\lambda \not\to [\lambda]^2_\kappa$;

(2) non-productivity of λ-chain conditions for Boolean algebras.

Proving instances of $Pr_l(\lambda, \lambda, \kappa, \theta)$ uses pcf theory, giving absolute theorems.

QUESTION (Shelah [564]): Is the following consistent: ZFC + $\aleph_\omega > 2^{\aleph_0} \to [\aleph_1]^2_3$?

QUESTION (Shelah [564]): Is the following consistent: ZFC + $2^{\aleph_0} \geq \aleph_2$ and if $\mathbf{P}$ has the $2^{\aleph_0}$-c.c. and $\mathbf{Q}$ has the $\aleph_2$-c.c., then $\mathbf{P} \times \mathbf{Q}$ has the $2^{\aleph_0}$-c.c.

Was Sierpinski right? Although this question seems to be still on the menu (Shelah 1996, and Shelah [546]), you'd better order quickly:

Cardinal Arithmetic, page 156:

"The appetite comes with the eating."

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