# The Vaught Conjecture: a counterexample

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## The Topological Vaught Conjecture

A continuous action of a Polish topological group on a Polish space has finitely many, countably many, or $2^{\aleph_0}$, orbits.

(D. E. Miller, 1980)

## The Vaught Conjecture

Let $T$ be a (complete) theory in a countable language of first-order predicate calculus. Then $T$ has finitely many, countably many, or $2^{\aleph_0}$, distinct countable models.

(R. Vaught, 1961)

VC is false for

- second-order logic
- $\mathcal{L}_{\omega_1,\omega}$-theories.

VC is true for $\omega$-stable theories. (Shelah)

It is unknown for stable theories.

TVC is true for

- locally compact groups

- Abelian groups

The case where the group is $\mathrm{Sym}(\omega)$ is canonical.

Let $L$ be a countable language of first-order predicate calculus.

Let $\mathcal{Q}$ be the set of all questions we could ask about an $L$-structure with domain $\omega$. (That is, questions about the truth of all the countably many atomic formulae.)

Associate each element $f$ of ${}^{\mathcal{Q}}\{YES, NO\}$ with an $L$-structure $\mathfrak{M}_f$.

If $T$ is a set of $\forall\exists$ formulae, then

$$\{f : \mathfrak{M}_f \models T\}$$

is $G_\delta$.

**Theorem** (Morley, 1970) Let $T$ be a (complete) theory in a countable language of first-order predicate calculus. Then $T$ has finitely many, countably many, $\aleph_1$, or $2^{\aleph_0}$, distinct countable models.

Let $T$ be a theory (set of formulae).

$p$ is a *complete $n$-ary type* iff it is a set of formulae $\phi(x_1, \ldots, x_n)$, such that there exist $\mathfrak{M}$ and $m_1, \ldots, m_n \in \mathfrak{M}$ such that

$$p = \{\phi(x_1, \ldots, x_n) : \mathfrak{M} \models \phi(m_1, \ldots, m_n)\}.$$

Let $L = L_0$.

Create $L_{\alpha+1}$ from $L_\alpha$ by adding $\bigwedge p$ whenever $p$ is a complete $n$-ary type in $L_\alpha$ compatible with $T$.

At limits, take unions.

There are three cases.

(1) For some $\alpha < \omega_1$, the space of complete types over $L_\alpha$ is uncountable. —$2^{\aleph_0}$ countable models.

(2) All spaces countable, and eventually constant. —at most $\aleph_0$ countable models.

(3) We obtain a non-discrete, scattered tree of scattered height $< \omega_2$. —$\aleph_1$ countable models.

## Building an example

Create predicate letters to say "$\|\langle x_1, \ldots, x_n\rangle\| > -\infty$".

Add axioms that say

(1) "If $\|\langle x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\rangle\| > -\infty$, then $\|\langle x_1, \ldots, x_n\rangle\| > -\infty$"; and

(2) "If $\|\langle x_1, \ldots, x_n\rangle\| > -\infty$, and $\sigma$ is a permutation, then $\|\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)}\rangle\| > -\infty$".

Define a "rank function" $\|.\|$ with values in $\{-\infty\} \cup \mathbf{On} \cup \{\infty\}$.

Let $\mathfrak{M}$ be a model of the theory. Define its *spectrum* to be

$$\text{spec}\mathfrak{M} = \{\|\langle x_1, \ldots, x_n \rangle\| : x_1, \ldots, x_n \in \mathfrak{M}\}.$$

The spectrum is one of $\{-\infty\} \cup \alpha$ or $\{-\infty\} \cup \alpha \cup \{\infty\}$, where $\alpha$ is an ordinal.

For countable models, we aim to achieve the following:

- There are $\aleph_1$ spectra.

- The spectrum is (as nearly as possible) categorical.

Add predicate letters $P^n_{\mathcal{F}}(x_1, \ldots, x_n)$, asserting that on the set of subsequences of $\langle x_1, \ldots, x_n \rangle$, the function $\|.\|$ is constrained to belong to some particular set $\{\|.\|_i : i \in I\}$.

Add axioms saying that this information is

- Relevant.

- Consistent.

- True.

Add further axioms imposing homogeneity on a model.

**Major problem:** How can we describe ordinal-valued functions $\|.\|_i$ in a first-order way?

If we can express the concept $\|\mathbf{x}\| = \alpha$ (for $\alpha$ a countable ordinal), by some type $F(\mathbf{x}, \alpha)$, then we end up with *non-standard ordinal labels* $F(\mathbf{x}, \alpha)$ where $\alpha$ is a non-standard ordinal. And there are too many non-standard ordinals.

If $\mathfrak{M} \models F(\mathbf{x}, \alpha)$, where $\alpha$ is non-standard, then certainly $\|\mathbf{x}\| = \infty$.

So the problem is: make sure the $\|.\| = \infty$ region is as homogeneous as possible.

*Lose information* about $\mathbf{x}$ with $\|\mathbf{x}\| = \infty$, but *retain information* about all other tuples.

How?

**Basic approach:** We preserve information below $\|.\| = \infty$ by encoding information about small tuples of high rank $\|.\|$ in terms of large tuples of low rank.

For standard ordinal ranks, all the information will eventually appear in the region directly first-order accessible, namely immediately above $\|.\| = -\infty$.

We try to arrange that information is permanently lost in the non-well-founded region. But we run into a number of problems.

**Problem:** Limits are special.

**Solution:** "Compression," and the ordered structure $\Omega$.

**Problem:** We wish to have "partial descriptions" of arbitrarily large portions of the model.

**Solution:** "Arity suppression."

**Problem:** We wish to be able to amalgamate our descriptions of portions of the model.

**Solution:** Trees.

## Finiteness

It's convenient if, when we restrict the language to the set of letters $P^n_{\mathcal{F}}$, there are only finitely many types of any given arity. This will cut down the number of types in the full language and therefore the number of models.

This motivates the concept of a *unitary tree*, which is a tree corresponding to a partial order on tuples.

**Problem:** Trees, and arity suppression, combine to conflict with this notion of finiteness.

**Attempted solution:** Sequences of unitary trees of different arities, with arrows between them.

**Problem:** The unitary tree approach does not survive the amalgamation process.

**Solution:** Ambiguity trees.

So, we define the predicate letters $P^n_{\mathcal{F}}$ as follows: Let $\mathcal{F}$ be a (reasonable) family of ambiguity trees. Then we have a corresponding letter $P^n_{\mathcal{F}}(x_0, \ldots, x_{n-1})$, whose intended meaning is: "The restriction of the function $\|.\|$ to the set subtuples of $\langle x_0, \ldots, x_{n-1}\rangle$ can be found the family $\mathcal{F}$." (That is, is coded by a branch of one of the trees in $\mathcal{F}$.)

Now we write down some axioms:

- *Embedding axioms:* All the information contained in the $P^n_{\mathcal{F}}$ is consistent.

- *Compactness axioms:* Every tuple is contained in a tuple satisfying one of the letters $P^n_{\mathcal{F}}$.

- *Order axioms:* The letters $P^n_{\mathcal{F}}$ have their intended meaning—at least to accuracy $\omega$ (sc. as regards values of $\|.\|$ less than $\omega$). [This is the best we can do at first order.]

- *Saturation axioms:* Any tuple satisfying a letter $P^n_{\mathcal{F}}$ can be extended to a tuple whose subtuples satisfy any compatible family of letters $P^{n'}_{\mathcal{F}'}$.

## There are at least $\aleph_1$ models

Let $\alpha$ be a countable limit ordinal.

We construct a model with spectrum $\{-\infty\} \cup \alpha$ by a standard argument.

We can use arity suppression to ensure the existence of enough kinds of tuple with the function $\|.\|$ restricted to lie in the right range.

## There are at most $\aleph_1$ models

We need to improve two of our axiom schemata beyond what is directly expressible in first order.

- *Order axioms* All the letters $P^n_{\mathcal{F}}$ have their intended meaning. ["Correctness"]

- *Saturation axioms* All tuples can be extended in any way consistent with the similarity type of the model (the spectrum, plus one extra two-valued parameter). ["Generalised Saturation"]

We do this by reflecting information "downwards".

We are then able to construct an isomorphism between two models with the same similarity type, using a back-and-forth argument.

# Questions

- Can this be done better (eg. more economically)?

- Other questions: eg. is there a stable counterexample?

- What can be said about the category of Scott spaces?