Fixed Point Logics on Finite and Infinite Structures

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Fixed Point Logics

Fixed Point Logics

extensions of first order logic with fixed point operators.

- Extending study of inductive definability.
- Extensively studied in finite model theory.
- Related to questions of computational complexity.

In this talk

- Survey results on fixed point logics on finite structures.
- Discuss expressive power of these logics on infinite structures.

LFP

Least fixed point logic

Close first order logic simultaneously under first order operations and

$$\mathbf{lfp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

where,

- R is a k-ary relation variable.
- $\mathbf{x}$ is a k-tuple of variables.
- $\mathbf{t}$ is a k-tuple of terms.
- R occurs only positively in φ.

The operator **lfp** binds all occurrences of R and variables in $\mathbf{x}$ that occur in φ, but not those in $\mathbf{t}$

Semantics

The semantics of the **lfp** operator are given by:

$$\mathcal{A} \models \mathbf{lfp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

if

$\mathbf{t}^{\mathcal{A}}$ is in the least fixed point of the monotone operator defined by $\varphi(R, \mathbf{x})$ on $|\mathcal{A}|^k$.

Example:

Let $\varphi(R, x, y)$ be $x = y \vee \exists z(E(x, z) \wedge R(z, y))$

Then, $\mathbf{lfp}_{R,xy}\varphi(u, v)$ is a formula in two free variables that defines the transitive closure of E.

For any formula $\varphi(\mathbf{x})$ *of* LFP,

the set

$$\{(\mathcal{A}, \mathbf{c}) \mid \mathcal{A} \models \varphi[\mathbf{c}]\}$$

is decidable in polynomial time.

Conversely, in the presence of a distinguished binary relation $<$ *interpreted as a linear order on all structures,:*

Every polynomial time query is definable in LFP.

(Immerman; Vardi 1982)

Normal Form

For every formula of LFP, there is an equivalent (*over finite structures*) one of the form:

$$\exists x \, \mathbf{lfp}_{R,\mathbf{y}} \varphi(x, \ldots, x). \tag{1}$$

where φ is first order.

In contrast, on

$$(\omega, <)$$

the interleaving of **lfp** and $\neg$ yields a strict hierarchy of expressive power.

A normal form like (1) holds whenever fixed points are not reached at limit ordinals.

Inflationary Fixed Points

IFP is similar to LFP except it has an operator

$$\textbf{ifp}_{R,\mathbf{x}}\varphi(\mathbf{t})$$

to denote that $\mathbf{t}$ is in the union of the sequence of relations:

$$\varphi_\alpha = \bigcup_{\beta<\alpha} \varphi(\varphi_\beta/R).$$

where,

$$\varphi(\varphi_\beta/R)$$

is the relation defined by φ when φ_β is fixed as the interpretation of R.

Note: R need not be positive in φ.

If φ defines a monotone operator, then

$$\mathbf{ifp}_{R,\mathbf{x}}\varphi$$

defines the same relation as

$$\mathbf{lfp}_{R,\mathbf{x}}\varphi$$

So, in general:

$$\mathsf{LFP} \leq \mathsf{IFP}$$

On finite structures:

$$\mathsf{LFP} = \mathsf{IFP}$$

(Gurevich–Shelah 1986).

On $(\omega, <)$ there are first order $\varphi(R)$ such that

$$\mathbf{ifp}_{R,\mathbf{x}}\varphi$$

is not obtained as the least fixed point of any *monotone* first order formula.

Yet,

$$\mathsf{LFP} = \mathsf{IFP}$$

on $(\omega, <)$.

On what structures is IFP more powerful than LFP?

Second Order Logic

A collection of finite structures is definable in Σ_1^1

if, and only if,

it is decidable in NP.

(Fagin 1974).

On finite structures:

$$\mathsf{LFP} = \mathsf{IFP} \leq \Delta_1^1.$$

either by Fagin's theorem or by a direct translation.

Translation

Translating $\mathbf{ifp}_{R,\mathbf{x}}\varphi(\mathbf{t})$:

$\forall(\exists)$ a linear order on k-tuples

$\forall(\exists)$ a sequence of relations R^i indexed by k-tuples

$$[R^0 = \emptyset \wedge \ R^{i+1} = R^i \cup \varphi(R^i)] \rightarrow (\wedge)\ \exists i\, \mathbf{t} \in R^i.$$

Without the restriction to finite structures:

$$\mathsf{LFP} \leq \mathsf{IFP} \leq \Delta_2^1$$

$\forall(\exists)$ a sequence of relations indexed by k-tuples
$\forall(\exists)$ a well ordering of k-tuples
. . .

This cannot be improved as

$$\Pi_1^1 \cup \Sigma_1^1 \leq \mathsf{LFP}$$

on $(\omega, <)$.

Partial Fixed Points

For a formula $\varphi(R)$,

(not necessarily monotone in R)

define the sequence of relations:

$$\begin{aligned} \varphi_0 &= \emptyset \\ \varphi_{i+1} &= \varphi(\varphi_i/R) \end{aligned}$$

Define:

$$\mathbf{pfp}_{R,\mathbf{x}}\varphi$$

to be the relation:

$$\begin{array}{ll} \varphi_m & \text{if } \varphi_m = \varphi_{m+1} \\ \emptyset & \text{if there is no such } m. \end{array}$$

In the presence of linear order, the PFP definable properties of finite structures are exactly the PSPACE computable ones.

IFP $=$ PFP if, and only if, P $=$ PSPACE

(Abiteboul–Vianu, 1992).

PFP inductions may require an exponential number of steps:

$$[R(x) \wedge \exists y(y < x \wedge \neg R(y))]\vee$$

$$[\neg R(x) \wedge \forall y(y < x \rightarrow R(y))]\vee$$

$$\forall y\, R(y).$$

The semantics of PFP does not easily extend to infinite structures.

- What happens at limit ordinals?
- Can an induction on a well ordering cycle through all subsets?

  Depends on well-ordering the power-set?

Non-Deterministic Fixed Points

Consider two first order formulas:

$$\varphi_0(R, \mathbf{x}) \text{ and } \varphi_1(R, \mathbf{x})$$

We define a sequence of relations φ^b on a *finite* structure, indexed by *binary strings* b.

$$\begin{aligned} \varphi^{\epsilon} &= \emptyset \\ \varphi^{b\cdot 0} &= \varphi_0(\varphi^b/R) \cup \varphi^b \\ \varphi^{b\cdot 1} &= \varphi_1(\varphi^b/R) \cup \varphi^b \end{aligned}$$

nfp$_{R,\mathbf{x}}(\varphi_0, \varphi_1)$ is then the relation

$$\bigcup_b \varphi^b$$

On ordered structures:

A property is definable as:

$$\exists x \, \mathbf{nfp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)(x, \ldots, x)$$

where φ_0 and φ_1 are first order

if, and only if, it is NP.

A property is definable in NFP (*closing under negations and* **nfp**) if, and only if, it is definable in second order logic.

Without linear order,

$$\mathsf{NFP} < \mathsf{SO}$$

still,

$$\mathsf{IFP} = \mathsf{NFP} \text{ if, and only if, } \mathsf{P} = \mathsf{NP}.$$

(Abiteboul, Vardi and Vianu)

Infinite Structures

For any ordinal α, a binary string b of length α is a map

$$b : \alpha \to \{0, 1\}.$$

For a pair of formulas $\varphi_0(R, \mathbf{x})$ and $\varphi_1(R, \mathbf{x})$, define:

$$\begin{aligned}
\varphi^{\epsilon} &= \emptyset \\
\varphi^{b\cdot 0} &= \varphi_0(\varphi^b/R) \cup \varphi^b \\
\varphi^{b\cdot 1} &= \varphi_1(\varphi^b/R) \cup \varphi^b \\
\varphi^{b} &= \bigcup_{c<b} \varphi^c \qquad \text{when } b \text{ is of limit ordinal length}
\end{aligned}$$

here, $c < b$ means that c is a proper initial segment of b.

Well Ordered Structures

On well-ordered structures:

$$\mathsf{NFP} = \mathsf{SO}$$

$\mathsf{NFP} \leq \mathsf{SO}$ holds always

by a direct translation (*viz.* IFP).

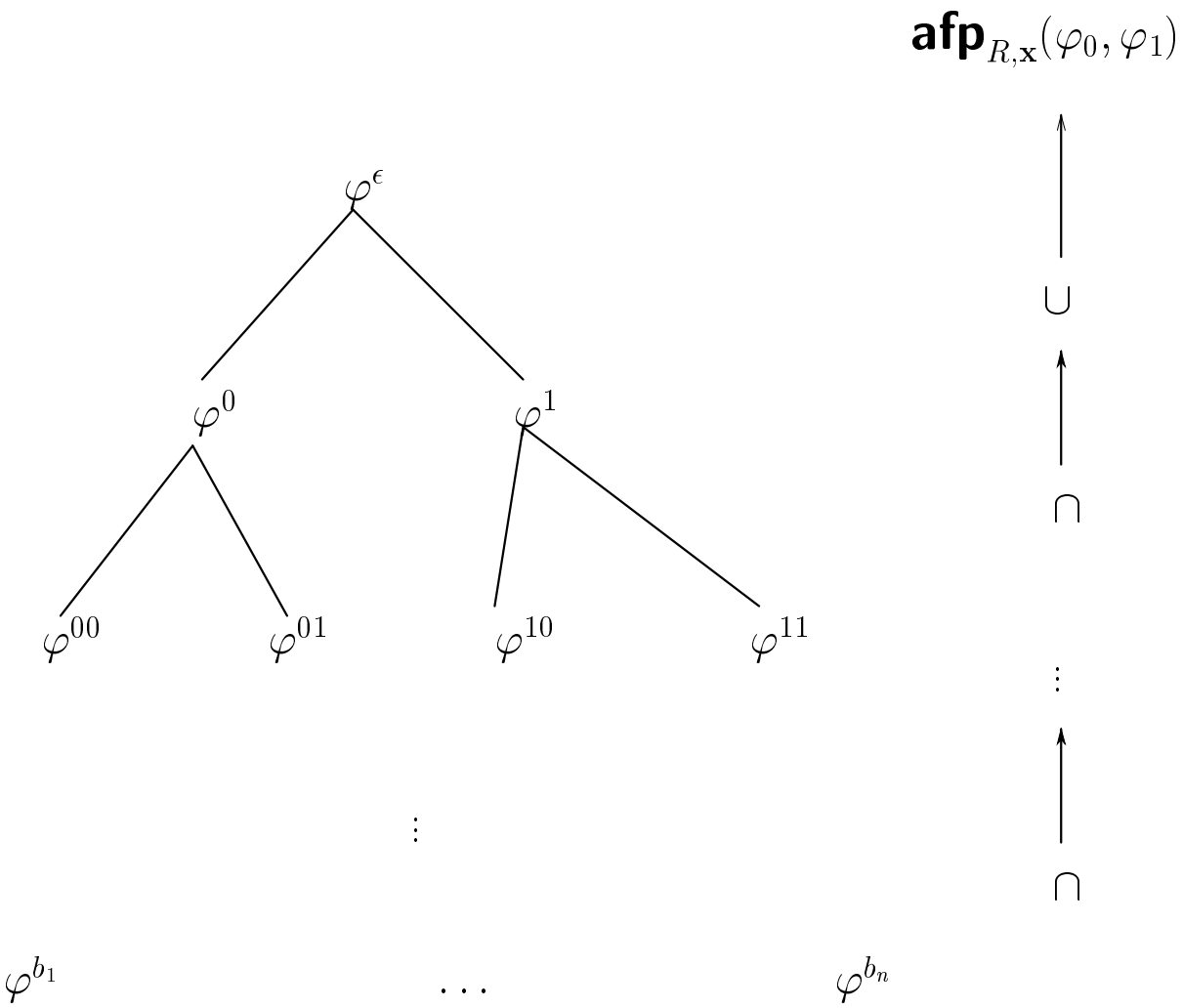
In the other direction, to translate $\exists R\, \theta$*,*

Intuitively, we can write φ_0 and φ_1 so that at each stage,

φ_0 adds the next element in the order to R.
φ_1 does not.

Beyond Σ^1_2, alternations of **nfp** and $\neg$ correspond exactly to *quantifier alternations*.

Alternating Fixed Point Logic



AFP = PFP

Games

Player 1 and Player 2 take turns in deciding which of the two formulas

$$\varphi_0 \text{ or } \varphi_1$$

to apply.

Player 1 plays at limit ordinals.

A tuple $\mathbf{x}$ is in

$$\mathbf{afp}_{R,\mathbf{x}}(\varphi_0, \varphi_1)$$

if Player 1 has a winning strategy for placing it in the constructed relation.

In what situations is this game determined?

Determinacy

There are (uncountable) structures for which there are formulas with undetermined games.

If a fixed point is always reached at ω, then the game is determined.

> The set of ω-length binary strings b such that $\mathbf{x} \in \varphi^b$ is necessarily an open set

What happens on countable structures where the game does not close at ω?