The random graph and its relations

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The random graph

Theorem 1 (Erdős and Rényi) *There is a countable graph R with the property that a random countable graph (edges chosen independently with probability $\frac{1}{2}$) is almost surely isomorphic to R.*

The graph R has the properties that

- it is *universal*: any finite (or countable) graph is embeddable as an induced subgraph of R;

- it is *homogeneous*: any isomorphism between finite induced subgraphs of R extends to an automorphism of R.

Universal homogeneous structures

Theorem 2 (Fraïssé) *R is the unique countable universal homogeneous graph.*

There are many other examples to which Fraïssé's theorem or variants apply:

- the random tournament, digraph, hypergraph, etc.;
- the universal total order $\mathbb{Q}$ (Cantor), partial order, etc.;
- the universal triangle-free graph (Henson), N-free graph (Covington), locally transitive tournament (Lachlan), two-graph, etc.;
- the universal locally finite group (Hall), Steiner triple system (Thomas), etc.

More generally . . .

There is a more general and powerful version due to Hrushovski. It constructs pseudoplanes, distance-transitive graphs, and examples related to sparse random graphs (among other things).

See the survey article by Wagner in Kaye and Macpherson, *Automorphisms of First-Order Structures*.

Constructions of R

Construction 1. Take any countable model of ZF, and join x to y if $x \in y$ or $y \in x$.

In fact we don't need all of ZF, only the null set, pairing, union, and foundation axioms. So the standard model of finite set theory (the set $\mathbb{N}$, with $x \in y$ if the xth binary digit of y is 1) gives an explicit construction (Rado).

Construction 2. Let $\mathbb{P}_1$ be the set of primes congruent to 1 mod 4. Join p to q if p is a quadratic residue mod q.

If we use instead $\mathbb{P}_{-1}$, the set of primes congruent to -1 mod 4, we obtain the random tournament.

First-order graph properties

The graph R 'controls' first-order properties of finite random graphs.

Theorem 3 *(*Glebskii *et al.) A first-order sentence in the language of graphs holds in almost all finite graphs if and only if it holds in R.*

In particular, there is a zero-one law for first-order sentences.

Of course, most interesting graph properties are not first-order!

Pigeonhole property

A structure X has the *pigeonhole property* if, whenever X is partitioned into two parts, one of the parts is isomorphic to X.

Theorem 4 *The countable graphs with the pigeonhole property are the complete graph, the null graph, and the random graph.*

Theorem 5 *The countable tournaments with the pigeonhole property are the random tournament, ordinal powers of ω, and their converses.*

Reducts

A subgroup of the symmetric group is *closed* in the topology of pointwise convergence if and only if it is the automorphism group of a first-order structure (which can be taken to be a homogeneous relational structure).

Theorem 6 (Thomas) *There are five closed subgroups of* Sym(R) *containing* Aut(R), *viz.* Aut(R), *the group of automorphisms and anti-automorphisms of* R, *the group of switching-automorphisms of* R, *the group of switching-automorphisms and anti-automorphisms of* R, *and* Sym(R).

Similar results are known in a few other cases, e.g. $\mathbb{Q}$ (as ordered set), random hypergraphs.

Reconstruction

The *perfect graph theorem* (Lóvasz) asserts that the complement of a perfect graph is perfect.

The P_4-*structure* of a graph is the collection of subsets which induce paths on four vertices. The *semi-strong perfect graph theorem* (Reed) asserts that a graph which has the same P_4-structure as a perfect graph is perfect.

Cameron and Martins proved that for almost all finite graphs G, the only graphs with the same P_4-structure as G are G and its complement. The analogous statement is true with any finite collection of finite graphs in place of $\{P_4\}$.

The proof uses Theorems 6 and 3, and some elementary model theory.

B-groups

A *B-group* is a group X with the property that any primitive permutation group G which contains the right regular action of X is doubly transitive.

Theorem 7 *For almost all n, every group of order n is a B-group.*

The proof uses the Classification of Finite Simple Groups.

By contrast, no countable B-groups are currently known.

Countable non-B-groups

A *square-root set* in a group X is a set of the form

$$\sqrt{a} = \{x : x^2 = a\}.$$

It is *non-principal* if $a \neq 1$.

Theorem 8 (Cameron and Johnson)
Suppose that the countable group X is not the union of finitely many translates of non-principal square-root sets. Then a random Cayley graph for X is isomorphic to R. Hence X is not a B-group.

Countable non-B-groups

Theorem 9 *There is no countable abelian B-group.*

Proof. Let $X_2 = \{x \in X : x^2 = 1\}$. If $|X : X_2|$ is infinite, then Theorem 8 applies. Otherwise, X has finite exponent and so $X = Y \times Z$ with Y and Z infinite; then X is contained in the primitive group $\mathrm{Sym}(\mathbb{N}) \wr \mathrm{Sym}(2)$ (with the product action).

Theorem 10 *A countable simple group with more than two conjugacy classes is not a B-group.*

Proof. Consider $\{x \mapsto a^{-1}xb : a, b \in X\}$.

Problems on B-groups

- Can one use other countable homogeneous structures to find further non-B-groups? What about Hrushovski's method?

- Are there any countable B-groups?

- Is the hypothesis of Theorem 8 necessary?

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