Local connectedness and distance functions.

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ABSTRACT: Local connectedness functions for $(\kappa, 1)$-simplified morasses, localisations of the coupling function c studied in [M96, §1], are defined and their elementary properties discussed. Several different, useful, canonical ways of arriving at the functions are examined. This analysis is then used to give explicit formulae for generalisations of the local distance functions which were defined recursively in [K00], leading to simple proofs of the principal properties of those functions. It is then extended to the properties of local connectedness functions in the context of κ-$\mathbb{M}$-proper forcing for sucessor κ. The functions are shown to enjoy substantial strengthenings of the properties (particularly the Δ-properties) hitherto proved for both c and for Todorcevic's ρ-functions in the special case $\kappa = \omega_1$. A couple of examples of the use of local connectedness functions in consort with κ-$\mathbb{M}$-proper forcing are then given.

Introduction.

In *The two cultures of mathematics* ([G]) Gowers argues extremely persuasively that although combinatorics appears to have fewer layers of theory through which one has to wade before one can prove, or even appreciate, significant results, one should not imagine that it is more superficial than other areas of mathematics. He points out that while in algebraic number theory one deep theorem will use another, which in turn uses a third ..., in combinatorial parts of mathematics there would often be no hope of proving one result if one was unaware of the general principles introduced in the proof of another, which in turn would appear to be unprovable if one was unaware of the main ideas of the proof of a third ..., even if there is no logical dependence between results. So to make progress in combinatorial areas one must command a wide collection of typical themes of arguments. These important ideas can often, or perhaps even usually, be pithily expressed as general principles of extensive applicability. This paper can be seen as a contribution to clarifying some new insightss about one such general principle, *stepping up.*

The idea behind stepping up is that when one can prove something about a collection of objects of a certain size and wants to prove an analogue for things of a larger size one can frequently try to use a similar proof techniques, reducing problems that arise about larger size objects to ones about smaller size ones. That both the finite version of Ramsey's theorem and Erdős, Hajnal and Rado's result ([EHR65]) that the Erdős-Rado theorem is optimal are proved using stepping up arguments evidences that they have a long and illustrious history in both finite and infinite combinatorics.

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While some stepping up arguments can be carried out 'by hand' it is often helpful, and sometimes necessary, to use *auxiliary apparatus*. A good, simple example of this is Erdős and Hajnal's proof ([EH]) that Chang's conjecture is equivalent to $\omega_2 \longrightarrow [\omega_1]^2_{\omega_1,<\omega_1}$. One uses a function $e : [\omega_2]^2 \longrightarrow \omega_1$ such that each $e(.,\alpha)$ is an injection of α into ω_1 to reduce questions about functions on a large set (the finite subsets of ω_2) to questions about functions on a smaller one (the finite subsets of ω_1) which one already knew could be dealt with sucessfully. Then, using the auxiliary aparatus again, one steps the results back up again.

Another interesting example from the same period is Baumgartner's proof ([B]) that it is consistent that there is family of subsets of ω_1 of size ω_2 which is *strongly almost disjoint*, that is, the intersection of any two of which is finite. This *generic stepping up argument* appears to be somewhat different as one does not step up a smaller s.a.d. family *per se*, but rather steps up a ccc forcing notion for adding a family of ω_2-many (disjoint mod-finite) subsets of ω to one for adding the desired s.a.d. family of subsets of ω_1. Nevertheless at the broadest level the structure of the proof is as before. The crux of the argument is showing that the stepped up forcing is ccc. There a family of ω_2-many subsets of ω_1, the intersection of any two of which is *countable*, is used to reduce the problem of how to limit the size of variuos collections of finite subsets of ω_1 that one has to deal with (to being countable) to a fact that one already knows: that the number of finite subsets of any countable set is countable.

However stronger tools having properties beyond those that can be proven to exist in ZFC alone have also proven so widely useful as to constitute basic ideas, or perhaps basic subideas, in themselves. Most of these tools can crudely be fitted into one of two collections, collections for which Velleman's simplified morasses and Todorčević ρ-functions are limning emblems.

The first type of tool is used in stepping up arguments in which something of cardinality greater than κ is built by an induction of length $\kappa + 1$. These tools are frameworks which enable one to amalgamate and fit together (through embeddings) pieces of size less than κ of some desired final object. At limit stages in the inductions, including the final one, limits are taken along some collection of functions more complicated than a simple chain in order to allow the final object to have size greater than κ. The power of these frameworks generally comes from their additional coherence properties. Examples of the tools from this circle of ideas include morasses, simplified morasses ([V84]), some uses of $\square_\kappa$, historicised forcing and pseudo- and semi-morasses ([SS], [K95], [M*3]), the uniform forcing axioms from [SHL], and, a little more distantly, the Abraham-Shore technique ([AS], M*1]).

The other type of tool is (something coded by) a function from $[\kappa^+]^2$ to κ, usually with some additional helpful properties. Families $\langle B_\alpha \,|\, \alpha < \omega_2 \rangle$ of almost disjoint sets (as in [B]) can be fitted under this rubric (by setting $b(\alpha,\beta) = \mathrm{ssup}(B_\alpha \cap B_\beta)$), as can various types of κ^+-Aronszajn tree and κ-Kurepa tree (with or without restrictions on the sorts of subtrees that they have). A landmark in the development of this line of thought was Roitman's paper [R] in which such a function was used to help prove that there is a ccc forcing to add a sort of superatomic Boolean algebra analogue of a Kurepa tree.

The crucial property of the function, which Roitman called the *new* Δ*-property*, is that for every uncountable disjoint family $\mathcal{A}$ of finite subsets of ω_2 and every $\delta < \omega_1$ there are some $a, b \in \mathcal{A}$ such that $\delta < e(\alpha,\beta)$ for every $\alpha \in a$ and $\beta \in b$. This property clearly implies the negation of Chang's conjecture since if $X \in [\omega_2]^{\omega_1}$ then $e``\{\, \{x\} \,|\, x \in X \,\}$ is uncountable, showing that e witnesses the negation of the Erdős-Hajnal partition calculus equivalent of Chang's conjecture discussed above.

(In fact it turns out that a Skolemisation argument shows that the existence of a function with the new Δ-property is also an equivalent of the negation of Chang's conjecture.) But Chang's conjecture is consistent with ZFC assuming the consistency of relatively mild large cardinals (an ω_1-Erdős cardinal), so the assumption of the existence of a function with the new Δ-property goes beyond what can be harvested from in ZFC alone. (Assuming of course, as there seems every reason to, that ω_1-Erdős cardinals are consistent.) Roitman's paper was particularly significant because it was the first time such a function was used in generic stepping up.

There are several separate ways in which Roitman's precedent has been followed up in order to attack other problems involving generic stepping up for which it seemed that functions with even stronger 'Δ'-like properties would be helpful. Interestingly, each of these various ways involves a piggy-backing of stepping up arguments: stepping up is used to obtain a tool, and the tool is then used to help with a further stepping up argument.

The principle direction, probably first pursued in [Veličk] and later exploited for example in [KM] and [K98a], is to use functions which had already been used for non-generic stepping up and which are extracted directly from coherent frameworks. The best known of these are Todorčević's function ρ ([T87]) defined from $\Box_\kappa$ and the coupling function c defined from a $(\kappa, 1)$-simplified morass and analysed in [M96]. Proofs that ρ and c have the new Δ-property when $\kappa = \omega_1$ are implicit in [Veličk] and [M96], respectively.

Another direction, followed in papers such as [BS], [Ra] and [Kom], is to use functions obtained by a historicised forcing as auxiliary aparatuses. However, as pointed out in [M96], the fact that the functions are obtained generically is not necessary in order to prove the properties these papers exploit: the historicised forcings have the side-effect of adding a simplified morass, the functions used are immediately definable from the c coming from this simplified morass as $f(\nu, \tau) = \{\, \xi < \nu \mid c(\xi, \nu) \leq c(\nu, \tau) \,\}$, and the properties used can all be derived for f defined in this way from the coupling function c of *any* simplified morass. (The property used in [BS] can also be proved from the properties of ρ if ρ is substituted for c in the definition of f and $\Box_{\omega_1}$ holds.)

Some people have gone beyond this and argue that, because it has been possible to go back to the definitions of ρ and c and prove further properties when necessary, ρ and c are inately superior to functions given as one-offs, for example through forcing. That argument seems tendentious unless backed by arguments that any stepping up aparatus that can be forced already exists in L and that ρ and/or c encapsulate all of the useful stepping up apparatus of L. While the study of ρ and c certainly has been very fruitful and while this paper itself focuses on related ideas, these two claims overstate the case.

There have been three principal sorts of development of the properties of ρ and c building on the new Δ-property, sometimes working in combination with one another. One of these is to improve the property to one that would facilitate prooof of the Knaster property for forcings rather than merely the countable chain condition. With this stronger unboundedness property one would be able to start with a family $\mathcal{A}$ of ω_1 many domains of conditions (finite subsets of ω_2) and some $\delta < \omega_1$ and thin the collection to some Δ-system $\mathcal{B}$ so that one still has ω_1 many domains of conditions and $\delta < e(\alpha, \beta)$ for all $\alpha \in a \setminus b$ and $\beta \in b \setminus a$ for any a, $b \in \mathcal{B}$. The point of this is that one can then thin $\mathcal{B}$ further to get a pair of conditions satisfying some other properties needed for compatibility, whereas the new Δ-property itself has to be the last thinning applied in any argument for the ccc. (See, for example, [K98a] for an argument of this type.) There are some problems with the published proof of this property for ρ (in [Bek]) and hitherto no proof for c has been given. This situation is remedied in [M*7].

Another evidently desirable generalisation is to get the analogue of the new Δ-property for arbitrary κ^+ instead of for ω_1. A primary use of this is to prove the κ^+-chain condition in generic stepping up arguments when the domains of conditions have size $< \kappa$ rather than being finite. This was done for c in [KM] and used to prove the analogue of Roitman's theorem for arbitrary successor cardinals.

The third sort of development is to strengthen the new Δ-property, for example, to ensure that given a Δ-system like $\mathcal{A}$ from two paragraphs ago one can find a pair of domains of conditions a, b such that $e(\alpha,\beta) \geq e(\alpha,\gamma)$ or $e(\beta,\gamma)$ for all triples $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$. As shown by a minute argument, given in [M*7], this is a strengthening of the new Δ-property under the mild assumption, certainly satisfied by all of the functions considered so far and a key to them being useful for stepping up, that $\{\xi \mid e(\xi,\zeta) \leq \delta\}$ is countable for all $\zeta < \omega_2$ and $\delta < \omega_1$. This variation has been used, for example, to give ccc forcings that add a Kurepa tree ([Veličk]) or a chain of length ω_2 in $\mathcal{P}(\omega_1)$ mod finite ([K98a]), and, implicitly, in adding superatomic Boolean algebras with Cantor-Bendixson rank $\omega_2 + 1$ and a countable collection of atoms at each stage of the Cantor-Bendixson process ([BS] and [Ra]).

As Chang's conjecture is not compatible with the existence of a superatomic Boolean algebra of the type added by Roitman, a Kurepa tree or a chain of length ω_2 in $\mathcal{P}(\omega_1)$ mod finite ([K98a]), none of these objects can be added by a ccc forcing without assuming the negation of Chang's conjecture, or, equivalently, the existence of a function with the new Δ-property. So in these cases, at least, one really *had* to find auxiliary apparatuses beyond ZFC in order to solve the problems at hand.

Some problems are even more unassailable, so much so that it is no good looking for an argument similar to the generic stepping ups considered so far but with an original twist on the properties of the stepping up function. For example, Koszmider ([K00]*) showed that if the continuum hypothesis holds there is no ccc forcing at all which adds a chain of functions of length ω_2 in $\omega_1^{\omega_1}$ ordered by domination mod finite. In order to solve such problems in *two-cardinal combinatorics* Koszmider had the extremely original, if in retrospect logical, idea of trying to step up forcings with finite domains to ones which, even though not ccc, have some variant of properness and so will preserve ω_1. Koszmider was inspired by his understanding of Todorčević's method of 'forcing with chains of models as side conditions' ([T85], [T89], K98b]) as a way of implementing the amalgamation arguments common in the proof of the countable chain condition in a more general setting.

Koszmider's insightful new method of $\mathbb{M}$-*proper forcing* is a melding of the two techniques of generic stepping up using auxiliary functions and forcing with chains, or more exactly, matrices of models as side conditions. Koszmider attaches finite sets of morass maps as 'side conditions' to conditions in a naive forcing, 'working parts.' Constraints by a single function, such as c, are replaced (implicitly) by constraints *synthetically* derived from these finite sets in the manner of proper forcing with matrices of models as side conditions. But properties of the simplified morass are used when carrying out amalgamation arguments, to prove $\mathbb{M}$-properness or the ω_2-chain condition, to give information about the relationship between constraints in the conditions which one is aiming to amalgamate. This information goes far beyond what could be gleaned merely from their coming from matrices (of $\in$-chains). As arguments using properness and matrices of models as side conditions seem to be limited to producing objects of size ω_1, Koszmider's method, which shows how to build objects

* I am grateful to Piotr Koszmider for making a preprint of [K00] available to me in early 2000.

of size ω_2, represents a considerable advance.

In §1 of this paper I analyse local versions of the coupling function c. These local versions are derived from a $(\kappa, 1)$-simplified morass in a similar way to c but locally, using collections of maps, rather than globally, using all of the morass maps. As the local functions are defined from chains of couplings perhaps it is reasonable to call them connectness functions and imagine that this was how c got its name as well! As well as proving some basic facts about these functions I investigate a couple of canonical ways of defining them. These canonical definitions give one a more concrete grasp on the functions than their abstract definition.

In §2 I continue the analysis of these local connectedness functions. Two further canonical ways of defining them are given. In §3 the analysis of these canonical definitions of the local connectedness functions is used to give explicit formulae for local distance functions. Local distance functions are another very interesting new notion Koszmider introduced in [K00] giving a recursive definition. The explicit formulae again give a more concrete grasp on the distance functions and make analysis and use of them easier. For example, the triangle equality for local distance functions is an immediate corollary of the explicit formulae, as are some remarks on when different localisations give rise to the same distances.

It is perhaps not surprising, given what happens when constraints are given *a priori*, that one often needs analogues for local connectedness functions of, for example, the variant of the new Δ-property used by [Velička] and [BS], in order to ensure that the amalgamations necessary to prove κ-$\mathbb{M}$-properness can be carried out. One also needs to understand how the different local connectedness functions derived from the sets of maps from pairs of conditions one is trying to amalgamate are related.

Thus, in §4 I define κ-$\mathbb{M}$-properness and build on the analysis of §1 to give generally applicable results of this sort, including analogues of the properties of c and ρ which have already proven so useful. In fact (the conclusions of) these analogues are stronger than the known results for c and ρ. The coherence properties of the simplified morass also ensure that the finite collections of maps from different conditions fit together in a compatible way and this gives one a reasonable chance of proving that κ^+ is preserved (often by showing that the κ^+-chain condition holds).

In §5, at the behest of the referee, I give two applications of the method, one topological, to scattered spaces of size κ^+, and one combinatorial. The reader is refered to that section for the details.

It may seem paradoxical to devote time to abstract analysis of these techniques after having started by highlighting a defence of the depth of combinatorial mathematics which emphasises its accumulation of general principles for solving problems rather than on theory-building within it. However, that would be to confound this defence with the heroic fallacy that, although general heueristics may be useful, one can always solve combinatorial problems *ab initio*.

Actually, sometimes it is simply *useful* to have information about the abstract ways in which general principles have and/or can be implemented rather than reinventing the wheel on each occasion. Although Gowers ([G]) emphasises that not all general principles can be precisely formulated, he also points out that some can be, and have then been “applied again and again.”

In this case it proved possible and, moreover, convenient to give proofs of a variety of different results in §5 below, [M*4], [M*5], [M*6] and [M*8] in a more or less uniform way using the facts proven here in §§1-4. As these facts are independent of the specfics of the various forcings, it seems clearer to present them separately. But it would be an anathema to have aimed to set out the one true way of κ-$\mathbb{M}$-proper forcing (using local connectedness functions), and I hope that there remains plenty of scope for productive variations to be devised, especially regarding §4.

After extolling the fruitfulness of these generic stepping up techniques let me close this discussion on a downbeat note. Unfortunately κ-$\mathbb{M}$-proper forcing does not seem to open the way to analogous constructions of objects of size greater than κ^+. In particular, there seems to be no reasonable analogue of κ-$\mathbb{M}$-proper forcing for $(\kappa, 2)$-simplified morasses and there are technical reasons concerning $(\kappa, 2)$-simplified morasses why it seems unlikely such analogues will found, even when $\kappa = \omega_1$. The general question of generically stepping up forcings with working parts of size less than κ^- (the cardinal predecessor of κ) to get objects of size at least κ^{++} remains an extremely important and radically open problem area.

For the most part the set theoretic notation used is standard. One piece of notation worth revising is that if $X \subseteq \mathrm{On}$ then $\mathrm{ssup}(X)$, the *strong supremum of* X, is the least ordinal α such that $X \subseteq \alpha$. Another is that $[\nu, \tau)$ is the interval of ordinals $\{\xi \mid \nu \leq \xi < \tau\}$, and similarly for (ν, τ) and $[\nu, \tau]$. A third is that if κ is a successor cardinal then κ^- is the cardinal predecessor of κ.

Definitions, Theorems, Lemmas and so on are numbered separately in each section. Thus, for example, Definition (8) of §1 is referred to as Definition (8) within §1 and as Definition (1.8) elsewhere.

I conclude this introduction by recalling some well-known definitions and facts about gap-one simplified morasses. I include give some sketches which may help comprehension. Similar sketches litter the remainder of the paper, but the reader may find it useful to supply more of their own to help their visualisation. The reader can consult [V84] and [M96] for more extensive expositions of the properties of gap-one simplified morasses, although this paper is, I hope, self-contained.

Definition 1. Let κ be a regular cardinal. $\mathbb{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ *is a* $(\kappa, 1)$-*simplified morass* if $\langle \theta_\alpha \mid \alpha < \kappa \rangle$ is an increasing sequence of ordinals less than κ and $\theta_\kappa = \kappa^+$,

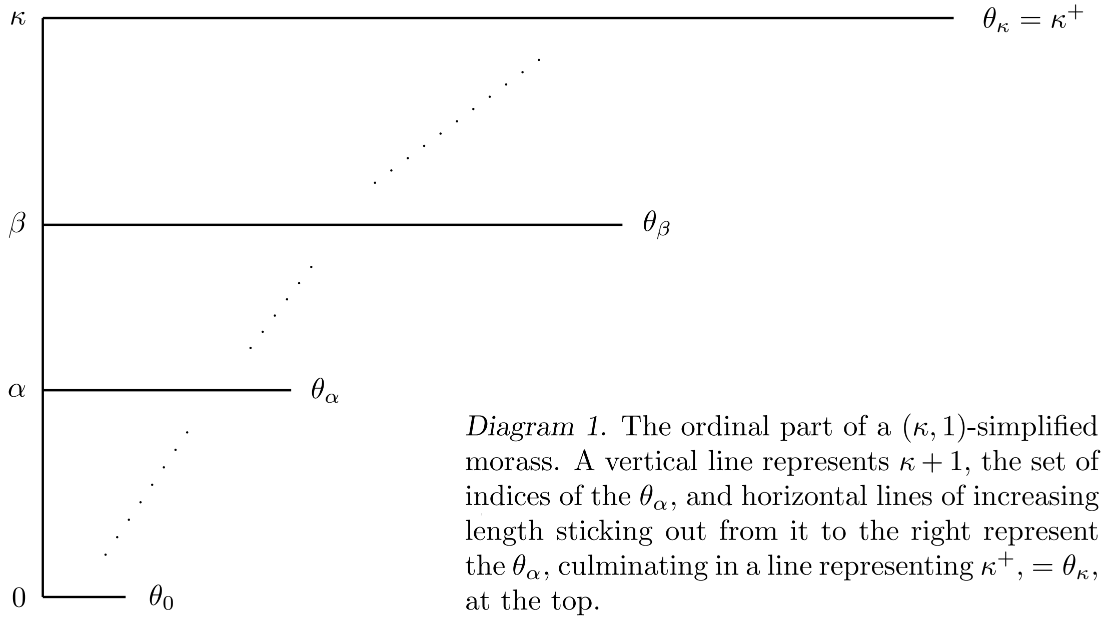

Diagram 1. The ordinal part of a $(\kappa, 1)$-simplified morass. A vertical line represents $\kappa + 1$, the set of indices of the θ_α, and horizontal lines of increasing length sticking out from it to the right represent the θ_α, culminating in a line representing κ^+, $= \theta_\kappa$, at the top.

and each $\mathcal{F}_{\alpha\beta}$ is a collection of maps from θ_α to θ_β

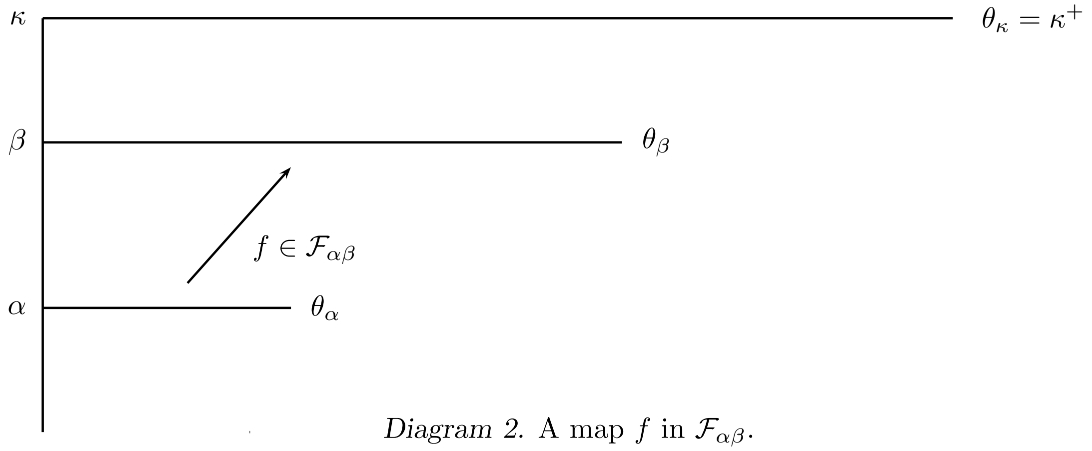

Diagram 2. A map f in $\mathcal{F}_{\alpha\beta}$.

such that the following properties hold:

(i) $\forall \alpha \leq \kappa \;\; \mathcal{F}_{\alpha\alpha} = \{\mathrm{id}\}$

(ii) $\forall \alpha \leq \beta \leq \gamma \leq \kappa \;\; \mathcal{F}_{\alpha\gamma} = \{ g \cdot f \mid f \in \mathcal{F}_{\alpha\beta} \;\; \& \;\; g \in \mathcal{F}_{\beta\gamma} \}$

(iii) $\forall \alpha < \kappa \; \big(\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}\} \;\; \& \;\; \theta_{\alpha+1} = \theta_\alpha + 1 \;\text{ or }\; \mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id},\, h\} \;\; \& \;\; \exists \sigma < \theta_\alpha \;\; \&$
$h \restriction \sigma = \mathrm{id} \;\; \& \;\; \forall \tau \; (\sigma + \tau < \theta_\alpha \longrightarrow h(\sigma + \tau) = \theta_\alpha + \tau) \big)$

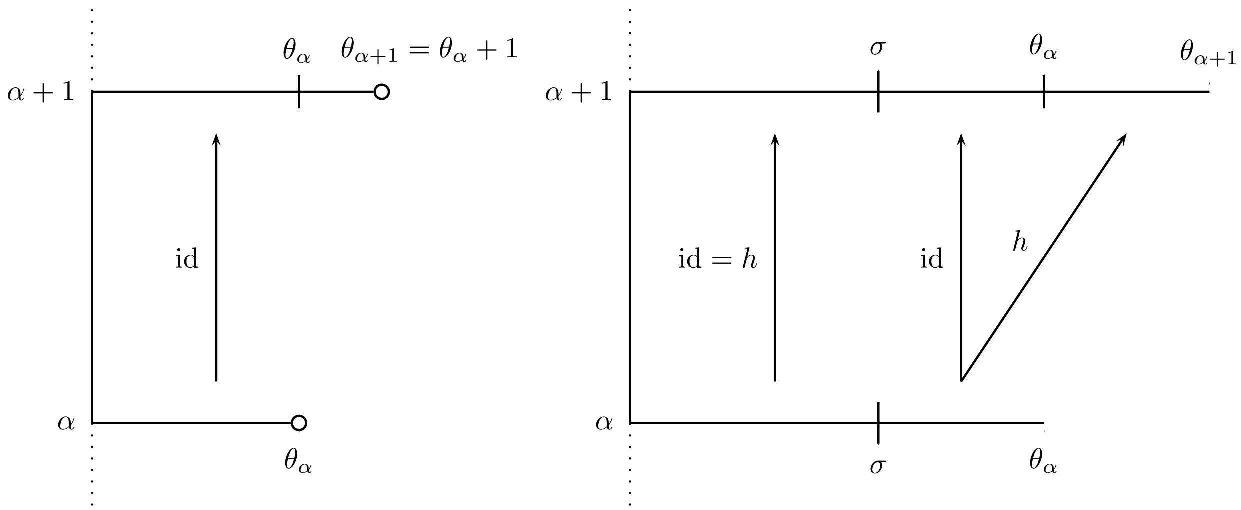

Diagram 3. Illustrating Definition (1.iii): for all $\alpha < \kappa$

$\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}\}$ and $\theta_{\alpha+1} = \theta_\alpha + 1$ or $\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}, h\}$, where $h \restriction \sigma = \mathrm{id} \restriction \sigma$ and $h(\sigma + \xi) = \theta_\alpha + \xi$ for all $\sigma_+\xi < \theta_\alpha$.

(iv) $\forall\alpha \leq \kappa\ \big(\alpha \text{ is a limit ordinal } \longrightarrow \forall\beta_0,\ \beta_1 < \alpha\ \forall f_0 \in \mathcal{F}_{\beta_0\alpha}\ \forall f_1 \in \mathcal{F}_{\beta_1\alpha}$
$\exists\gamma \in [\beta_0 \cup \beta_1, \alpha)\ \exists h \in \mathcal{F}_{\gamma\alpha}\ \exists g_0 \in \mathcal{F}_{\beta_0\gamma}\ \exists g_1 \in \mathcal{F}_{\beta_1\gamma}\ (f_0 = h \cdot g_0\ \&\ f_1 = h \cdot g_1)\big)$

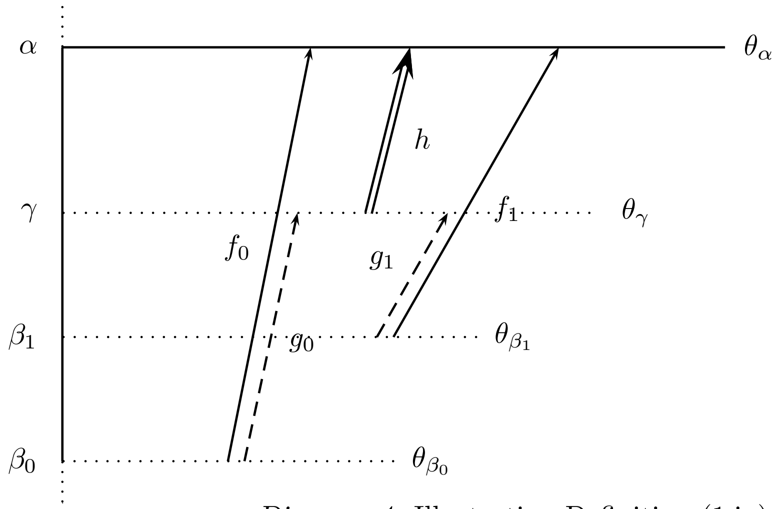

Diagram 4. Illustrating Definition (1.iv)

(v) $\bigcup\{\, f``\theta_\alpha \mid \alpha < \kappa\ \&\ f \in \mathcal{F}_{\alpha\kappa}\,\} = \kappa^+$.

Definition 2. Let $\mathbb{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa\rangle\rangle$ be a $(\kappa,1)$-simplified morass. Then $\mathcal{F} = \bigcup\{\,(\alpha, f) \mid \alpha < \kappa\ \&\ f \in \mathcal{F}_{\alpha\kappa}\,\}$.

The following well-known, fundamental fact due to Velleman always comes into play when dealing with simplified morasses.

Fact 3. (Velleman, [V84, Lemma 3.2]) For any $\alpha \leq \beta \leq \kappa$, f, $g \in \mathcal{F}_{\alpha\beta}$ with $\nu \in \mathrm{rge}(f) \cap \mathrm{rge}(g)$

there is some $\overline{\nu} < \theta_\alpha$ such that $f(\overline{\nu}) = \nu = g(\overline{\nu})$ and $f \restriction \overline{\nu}+1 = g \restriction \overline{\nu}+1$.

Proof. By induction on β for each α.

Another two useful facts from [V84] are the following.

Fact 4. (Stanley, [V84, Theorem 3.9]) If $\alpha \leq \beta \leq \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $\nu < \theta_\alpha$ then there is some $g \in \mathcal{F}_{\alpha\beta}$ such that $g \restriction \nu = f \restriction \nu$ and $g(\nu + \xi) = \mathrm{ssup}(g``\nu) + \xi$ for $\nu + \xi < \theta_\alpha$.

Proof. Again by induction on β for each α.

Fact 5. (Velleman, [V84, Corollary 3.5]) If $\langle f_i \mid i < \chi \rangle$ is a collection of maps with each $f_i \in \mathcal{F}_{\alpha_i\beta}$ for $i < \chi$ and $\chi < \mathrm{cf}\,(\beta)$ then there is some $\alpha \in [\sup(\{\alpha_i \mid i < \chi\}, \beta)$, some $f \in \mathcal{F}_{\alpha\beta}$ and maps $g_i \in \mathcal{F}_{\alpha_i\alpha}$ for all $i < \chi$ such that $f_i = f \cdot g_i$ for all $i < \chi$. In particular any collection of fewer than κ maps in $\mathcal{F}$ can be factored through some single common map $(\alpha, f) \in \mathcal{F}$.

The following notation is now more or less standard for simplified morasses (see, for example, [M96]).

Definition 6. Let $\alpha \leq \beta \leq \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $f(\overline{\nu}) = \nu$. Then $\psi_{\langle\alpha,\overline{\nu}\rangle,\langle\beta,\nu\rangle}$ is the function $f \restriction \overline{\nu}+1$. By Fact (3) this is a good definition. If $\beta = \kappa$ write ψ^α_ν for $\psi_{\langle\alpha,\overline{\nu}\rangle,\langle\beta,\nu\rangle}$. Write ν_α for the $\overline{\nu}$ such that $\psi^\nu_\alpha(\overline{\nu}) = \nu$.

Definition 7. If $\beta \leq \kappa$ and $\nu < \tau < \theta_\beta$ then $c_\beta(\nu, \tau) =$ the least $\alpha \leq \beta$ such that there is some $f \in \mathcal{F}_{\alpha\beta}$ with ν, $\tau \in \mathrm{rge}(f)$. Write c for c_κ and note that $c(\nu, \tau) < \kappa$ for all ν, $\tau < \kappa^+$ by properties (iii) and (v) of gap-one simplified morasses. c is the *coupling function* for $\mathbb{M}$.

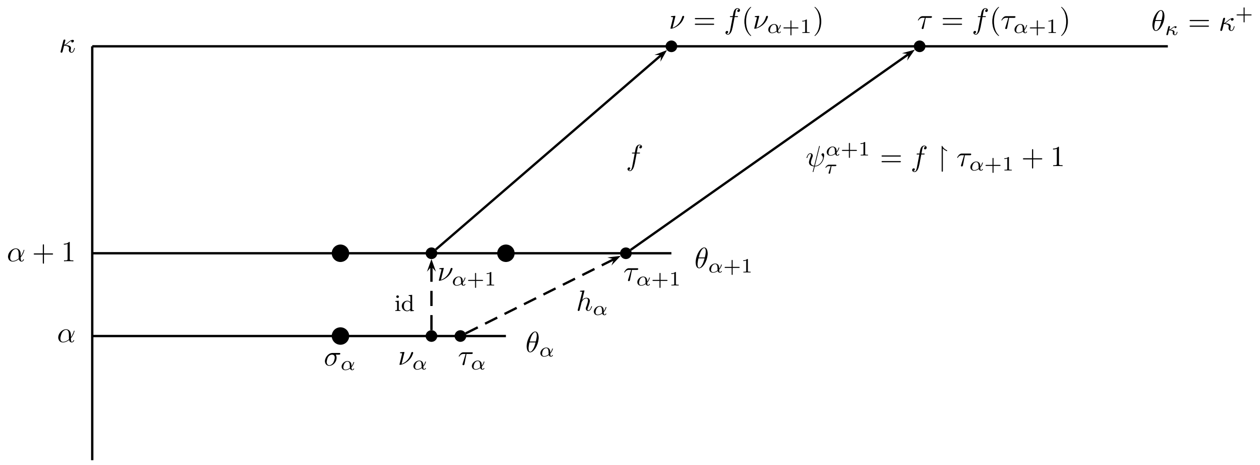

Diagram 5. Illustrating $c(\nu, \tau) = \alpha + 1$ being witnessed by $f \in \mathcal{F}_{\alpha+1\kappa}$. *cf.* Definitions (6) and (7).

As motivation for some of the properties of functions defined in §1 I also recall the following fact.

Fact 8. ([M96]) c has the following properties:

If $\alpha < \kappa$ and $\tau < \kappa^+$ then $\overline{\overline{\{\xi \mid c(\xi, \tau) \leq \alpha\}}} < \kappa$.

c is *subadditive*: if $\mu < \nu < \tau < \kappa^+$ then

(i) $$c(\mu, \nu) \leq \max(\{c(\mu, \tau), c(\nu, \tau)\})$$

(ii) $$c(\mu, \tau) \leq \max(\{c(\mu, \nu), c(\nu, \tau)\})$$

If $\xi < \lambda = \bigcup \lambda < \mu$ then there is some $\zeta \in [\xi, \lambda)$ such that for all $\nu \in [\zeta, \lambda)$ one has $c(\nu, \lambda) \leq c(\nu, \mu)$.

Definition 9. Let $\nu < \tau < \kappa^+$ and suppose that $c_\kappa(\nu, \tau) \leq \alpha$. Then $D_\alpha(\nu, \tau) = \operatorname{otp}([\nu_\alpha, \tau_\alpha))$, *i.e.* $= \operatorname{otp}(\tau_\alpha \setminus \nu_\alpha)$. Let $D(\nu, \tau) = D_{c(\nu,\tau)}(\nu, \tau)$.

In §§4,5 the local distance functions will be used in cardinal preservation arguments. In this context it will be necessary that the simplfied morass is *stationary*, a notion now defined.

Definition 10. Let $\mathcal{L}$ be a first order language. If $\mathcal{A} = (A, \ldots)$ is an $\mathcal{L}$-structure then $|\mathcal{A}| = A$.

Definition 11. Let $\mathbb{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ be a $(\kappa, 1)$-simplified morass. $\mathbb{M}$ is *stationary* if $\{\operatorname{rge}(f) \mid \exists \alpha\ (\alpha, f) \in \mathcal{F}\}$ is stationary in $[\kappa^+]^{<\kappa}$.

Finally, in applications, *e.g.*, in §5, a couple more facts about simplified morasses are useful. The first is a simple, well-known extension of Fact (1.3).

Fact 12. Let $\alpha \leq \beta \leq \kappa$, and f, $g \in \mathcal{F}_{\alpha\beta}$. If ν, $\tau \leq \theta_\alpha$ and $\operatorname{ssup}(f``\nu) = \operatorname{ssup}(g``\tau)$ then $\nu = \tau$ and $f \restriction \nu = g \restriction \nu$. ▲

Proof. By induction on β for each α. (For ν, $\tau < \theta_\alpha$ this is an immediate corollary of Facts (5) and (6), without the necessity of a separate inductive proof.) ▲

In contrast the following Δ-system type proposition does not seem to have been noted previously. The salient point is that no cardinal arithmetic assumption is necessary.

Proposition 13. (A Δ-system lemma for morass maps). Let $\langle (\alpha_\eta, F_\eta) \mid \eta < \kappa^+ \rangle$ be a collection of distinct maps in $\mathcal{F}$. Then there is a (stationary) set $S \subseteq \kappa^+$ such that $\langle \operatorname{rge}(F_\eta) \mid \eta \in S \rangle$ forms a Δ-system. ▲

Proof. Let $S_\kappa = \{\gamma < \kappa^+ \mid \operatorname{cf}(\gamma) = \kappa\}$. By Fodor's lemma one may as well assume that there is some $\alpha < \kappa$ and a stationary set $S \subseteq S_\kappa$ such that $\alpha_\eta = \alpha$ for all $\alpha \in S$.

Next set $G(\eta) = \operatorname{ssup}(\operatorname{rge}(F_\eta) \cap \eta)$ for $\eta \in S$. As $\operatorname{cf}(\eta) = \kappa$ and $\operatorname{otp}(\operatorname{rge}(F_\eta)) = \theta_\alpha < \kappa$ for all $\eta \in S$, one has that $G(\eta) < \eta$ for every $\eta < \kappa^+$. By Fodor's lemma again one has that there is some $\tau < \kappa^+$ and some stationary $T \subseteq S$ such that $G(\eta) = \tau$ for all $\eta \in T$.

Hence, for each $\eta \in T$ one has that there is some $\tau_\eta \leq \theta_\alpha$ such that $\tau = \operatorname{ssup}(F_\eta``\tau_\eta)$. By Fact (7) one has that $\tau_\eta = \tau_\xi$, $= \overline{\tau}$, say, and that $F_\eta \restriction \overline{\tau} = F_\xi \restriction \overline{\tau}$ for all η, $\xi \in T$. Consequently $\operatorname{rge}(F_\eta) \cap \eta = \operatorname{rge}(F_\xi) \cap \xi$ $(\subseteq \tau)$ for all η, $\xi \in T$. (It is by the use of Fact (7) here that the necessity for cardinal arithemetic assumptions is avoided.)

Note also that $\tau < \theta_\alpha$ as otherwise all of the maps F_η for $\eta \in T$ are identical. Finally, choose by induction, $\{\eta_i \mid i < \kappa^+\} \subseteq T$ by letting η_ζ be the least $\eta \in T$ such that $\bigcup \{\operatorname{rge}(F_{\eta_i}) \mid i < \zeta\} \subseteq \eta$. ▲

§1. On being 'connected in a set of maps' and the function c_A.

Let κ be a regular. As will be the case throughout the first four sections of the paper, suppose that $(\kappa, 1)$-simplified morasses exist. Fix $\mathbb{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa\rangle\rangle$ a $(\kappa, 1)$-simplified morass, and let $\mathcal{F}$ be as in Definition (I.2).

Definition 1. ([K00, Definition 2.9] for $\kappa = \omega_1$ and A finite.) Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. Then $\nu \leq \tau < \kappa^+$ are *α-connected in* A if and only if there are (α_0, f_0), $\ldots(\alpha_{k-1}, f_{k-1}) \in A$, and $\xi_k = \nu \leq \xi_{k-1} \leq \ldots \leq \xi_0 = \tau$ such that

$$\forall i < k\ (\alpha_i \leq \alpha \quad \& \quad f_i \in \mathcal{F}_{\alpha_i\kappa} \quad \& \quad \xi_i, \xi_{i+1} \in \mathrm{rge}(f_i)).$$

(If $\nu \neq \tau$ one may as well assume strict inequality holds in the sequence $\xi_0, \ldots\xi_k$.) $\langle\xi \mid i \leq k\rangle$ and $\langle(\alpha_i, f_i) \mid i < k\rangle$ are *witnesses* that ν and τ are α-connected in A.)

The key definition of this paper is the following definition of *local connectedness functions*.

Definition 2. Let $A \subseteq \mathcal{F}$. Define $c_A(\nu, \tau) =$ the least $\alpha < \kappa$ such that ν, τ are α-connected in A if ν, τ are α-connected in A for some $\alpha < \kappa$, and $= \kappa$ otherwise. Hence, for $\alpha < \kappa$, ν, τ are α-connected in A if and only if $c_A(\nu,\tau) \leq \alpha$. If $\langle\xi_i \mid i \leq k\rangle$ and $\langle(\alpha_i, f_i) \mid i < k\rangle$ are witnesses that $c(\nu, \tau) = \alpha$ then $\max(\{\alpha_i \mid i \leq k\}) = \alpha$.

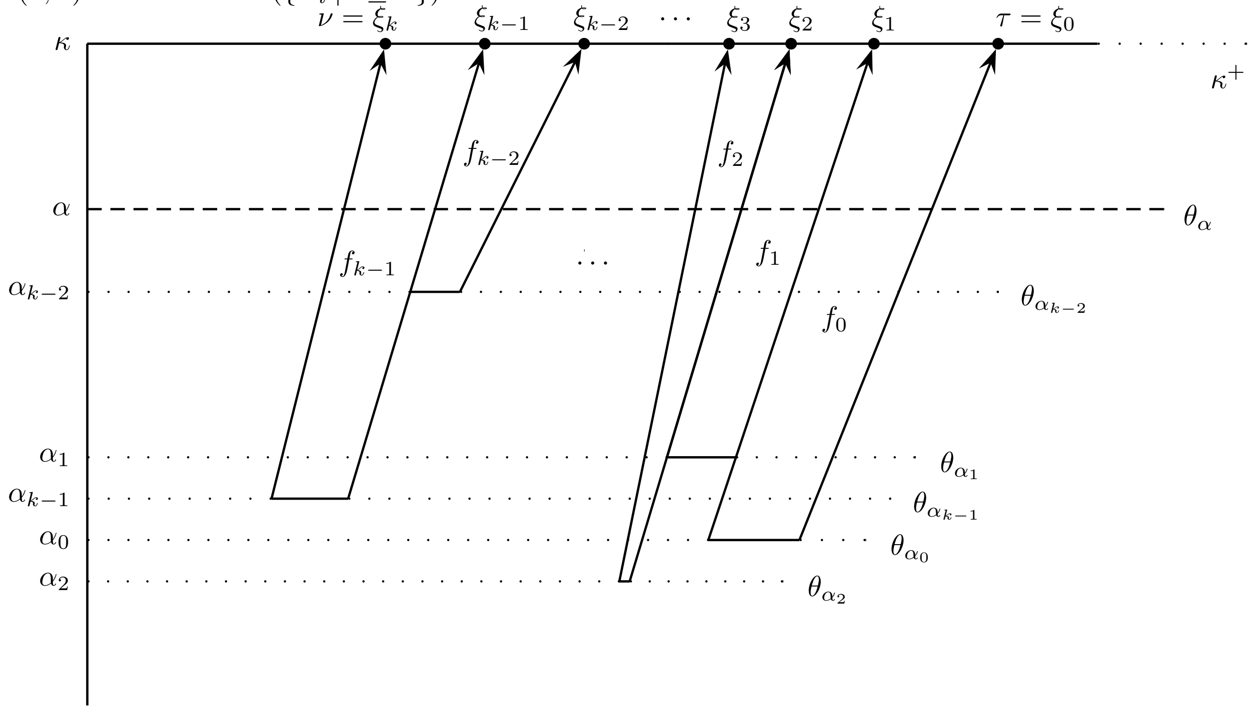

Diagram 6. Illustrating $c_A(\nu, \tau) \leq \alpha$, *cf.* Definitions (1) and (2), where $f_i \in \mathcal{F}_{\alpha_i\kappa}$, $(\alpha_i, f_i) \in A$ and ξ_i, $\xi_{i+1} \in \mathrm{rge}(f_i)$ for all $i < k$.

Lemma 3. $\forall A, B \subseteq \mathcal{F}\ \ \forall\nu < \tau < \kappa^+\ \ (A \subseteq B \longrightarrow c_B(\nu, \tau) \leq c_A(\nu, \tau))$

Proof. Immediate from the definitions of c_A and c_B.

The next lemma is fairly straightforward from the definition of the local connectedness functions and the basic facts about simplified morasses adumbrated in the introduction but will be used time and again.

Lemma 4. Suppose that $A \subseteq \mathcal{F}$ and $\nu < \tau < \kappa^+$. If $f \in \mathcal{F}_{\beta\kappa}$, $\tau \in \mathrm{rge}(f)$ and $c_A(\nu, \tau) \leq \beta$ then $\nu \in \mathrm{rge}(f)$.

Proof. Let $c_A(\nu, \tau) = \alpha$, say, with $\alpha \leq \beta$, and let $\nu = \xi_k < \ldots < \xi_0 = \tau$ and $\langle (\alpha_i, g_i) \,|\, i < k \rangle$ witness this. By induction on $i \leq k$ one has that $\xi_i \in \mathrm{rge}(f)$. For $\xi_0 = \zeta \in \mathrm{rge}(f)$. One also has $\alpha_i \leq \alpha \leq \beta$ so that that $g_i = h \cdot j$ for some $j \in \mathcal{F}_{\alpha_i\beta}$ and $h \in \mathcal{F}_{\beta\kappa}$ (by Definition (I.1.ii)). If one has that $\xi_i \in \mathrm{rge}(f)$ then this gives that $\xi_i \in \mathrm{rge}(f) \cap \mathrm{rge}(h)$, so, by Fact (I.3), there is some $\overline{\xi} < \theta_\beta$ such that $f(\overline{\xi}) = h(\overline{\xi}) = \xi_i$ and $f \restriction (\overline{\xi} + 1) = h \restriction (\overline{\xi} + 1)$. ($\overline{\xi} = (\xi_i)_\beta$ in the notation of Definition (I.5).) Then the fact that $\xi_{i+1} \in \mathrm{rge}(h)$ gives that $\xi_{i+1} \in \mathrm{rge}(f)$. Consequently $\nu = \xi_k \in \mathrm{rge}(f)$.

The next three lemmas show that Lemma (4) can be exploited to give very useful results about how local connectedness functions vary when one varies the parameter A even without any further understanding of the witnesses to connectedness.

Lemma 5. Suppose A, $B \subseteq \mathcal{F}$, $A \subseteq B$, and that for all $(\alpha, f) \in A$ and $(\beta, g) \in B \setminus A$ one has that $\alpha \leq \beta$. Suppose $\nu < \tau < \zeta < \kappa^+$. Then if $c_A(\nu, \tau) < \kappa$ and $c_A(\nu, \tau) \leq c_A(\tau, \zeta)$ one has that $c_B(\nu, \tau) \leq c_B(\tau, \zeta)$.

Proof. Let $\beta^* = \min(\{\beta < \kappa \,|\, \exists (\beta, g) \in B \backslash A\})$. Clearly as $c_A(\nu, \tau) < \kappa$ one has $c_B(\nu, \tau) = c_A(\nu, \tau)$ since none of the maps in $B \backslash A$ can help reduce $c_B(\nu, \tau)$. Similarly if $c_A(\tau, \zeta) < \kappa$ then $c_B(\tau, \zeta) = c_A(\tau, \zeta)$. On the other hand if $c_A(\tau, \zeta) = \kappa$ then $c_A(\nu, \tau) = c_B(\nu, \tau) \leq \beta^* \leq c_B(\tau, \zeta)$, since if $c_B(\tau, \zeta) < \kappa$ then one of any sequence of maps witnessing this is in $B \setminus A$.

Lemma 6. Suppose $\forall A \subseteq \mathcal{F}$, $A = B \cup \{(\beta, g)\}$, and $\alpha \leq \beta$ for all $(\alpha, f) \in A$. Suppose $\nu < \tau < \zeta < \kappa^+$ and that $\nu \in \mathrm{rge}(g)$ if $\tau \in \mathrm{rge}(g)$. Then $c_A(\nu, \tau) \leq c_A(\tau, \zeta)$ implies that $c_B(\nu, \tau) \leq c_B(\tau, \zeta)$.

Proof. If $c_A(\nu, \tau) < \kappa$ the lemma is immediate from Lemma (5). If $c_A(\nu, \tau) = \kappa$ and $c_B(\tau, \zeta) < \kappa$ then (β, g) is in any witnessing sequence for the latter, whence $c_B(\tau, \zeta) = \beta$ and $\tau \in \mathrm{rge}(g)$. By the hypothesis of the lemma this gives that $\nu \in \mathrm{rge}(g)$, so that $c_A(\nu, \zeta) = \beta$.

Note 7. $c = c_{\mathcal{F}}$.

Proof. Let $\alpha = c_{\mathcal{F}}(\nu, \tau)$. If $\nu < \tau < \kappa^+$ then $c_{\mathcal{F}}(\nu, \tau) \leq c(\nu, \tau)$ by the definition of $c_{\mathcal{F}}$. On the other hand, if $f \in \mathcal{F}_{\alpha\kappa}$ and $\tau \in \mathrm{rge}(f)$ then $\nu \in \mathrm{rge}(f)$ by Lemma (4), so $c(\nu, \tau) \leq c_{\mathcal{F}}(\nu, \tau)$. But there is some such f since $\tau \in \mathrm{rge}(g)$ for any $g \in \mathcal{F}_{\beta\kappa}$ with $\beta < \kappa$ the first map of a witnessing sequence to the value of $c_{\mathcal{F}}(\nu, \tau)$ and g factors as some $f \cdot h$ with $f \in \mathcal{F}_{\alpha\kappa}$ and $h \in \mathcal{F}_{\beta\kappa}$ by (I.1.ii). (Or, more simply, by the neatness of $\mathbb{M}$.)

Thus the functions c_A generalise the coupling function c and can be thought of as localisations of it. The remaining lemmas of this section and §§2-4 will demonstrate that the c_A share some of c's most important properties and so are good generalisations of it.

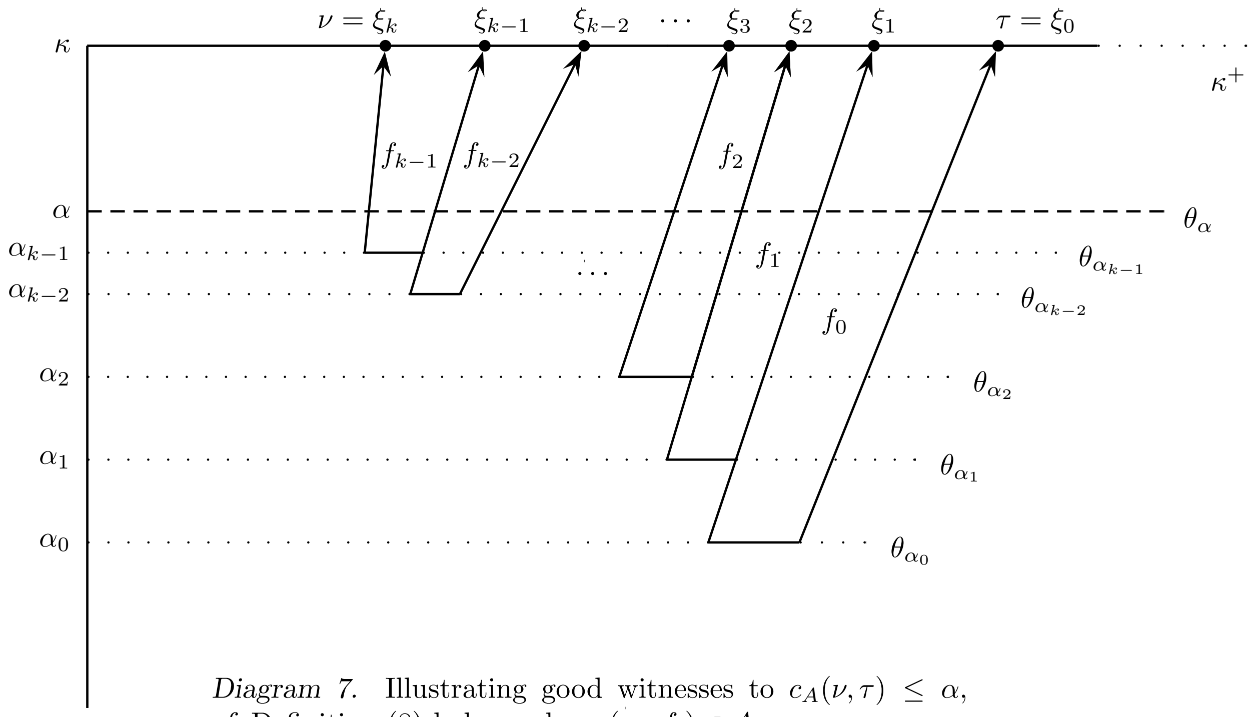

Diagram 7. Illustrating good witnesses to $c_A(\nu,\tau) \leq \alpha$, *cf.* Definition (8) below, where $(\alpha_i, f_i) \in A$.

The next lemma shows that one can assume without loss of generality that the sequence of witnessing maps to the value of any $c_A(\nu,\tau)$ come from increasingly high up in the morass.

Definition 8. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. Suppose that $\nu < \tau < \kappa^+$ and that $c_A(\nu,\tau) \leq \alpha$. Then $\langle \xi_i \mid i \leq k\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle$ are *good witnesses* that $c_A(\nu,\tau) \leq \alpha$ or *good witnessing sequences* to $c_A(\nu,\tau) \leq \alpha$ if they witness $c_A(\nu,\tau) \leq \alpha$ as in Definitions (1) and (3) and if $\langle \alpha_i \mid i < k\rangle$ is strictly increasing. They are *good witnesses that* $c_A(\nu,\tau) = \beta$ or *to the value of* $c_A(\nu,\tau)$ if they are good witnesses that $c_A(\nu,\tau) \leq \beta$ and $c_A(\nu,\tau) = \alpha_{k-1} = \beta$.

Lemma 9. Suppose that $A \subseteq \mathcal{F}$, $\alpha < \kappa$, $\nu < \tau < \kappa^+$ and $c_A(\nu,\tau) = \alpha$. Then there are good witnesses $\langle \xi_i \mid i \leq k\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle$ to the fact that $c_A(\nu,\tau) = \alpha$.

Proof. Let $\langle \xi_i \mid i \leq k\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle$ be any witnesses that $c_A(\nu,\tau) = \alpha$. Suppose $\alpha_{i+1} \leq \alpha_i$. One has that $c_A(\xi_{i+2}, \xi_{i+1}) \leq \alpha_i$, since $\xi_{i+2}, \xi_{i+1} \in \mathrm{rge}(f_{i+1}))$, and $\xi_{i+1} \in \mathrm{rge}(f_i)$, by the definition of a witnessing sequence. Hence, by Lemma (4), one has that $\xi_{i+2} \in \mathrm{rge}(f_i)$. Thus the sequences generated by iteratively omitting every pair (α_{i+1}, f_{i+1}) and ξ_{i+1} such that $\alpha_{i+1} \leq \alpha_i$ are also witnesses that $c_A(\nu,\tau) = \alpha$

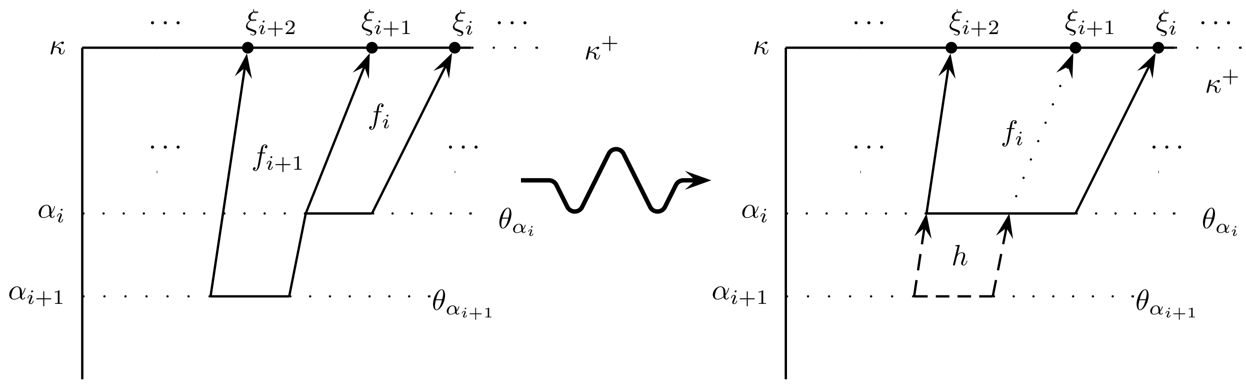

Diagram 8. Illustrating the induction step towards obtaining good witnesses in Lemma (9). As $f_{i+1} \in \mathcal{F}_{\alpha_{i+1}\kappa}$ there is some (unique) $g \in \mathcal{F}_{\alpha_{i+1}\alpha_i}$ and some $h \in \mathcal{F}_{\alpha_i\kappa}$ such that $f_{i+1} = h{\cdot}g$. Since $\xi_{i+1} \in \mathrm{rge}(f_i) \cap \mathrm{rge}(f_{i+1})$ one has that $h \restriction (\xi_{i+1})_{\alpha_i}+1 = f_i \restriction (\xi_{i+1})_{\alpha_i}+1$, so $\xi_{i+2} \in \mathrm{rge}(f_i)$.

Note 10. Suppose $A \subseteq \mathcal{F}$, $\alpha < \kappa$, $\nu < \tau < \kappa^+$ and $c_A(\nu, \tau) = \alpha$. If $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\alpha_i, f_i) \mid i < k \rangle$ are witnesses to the fact that $c_A(\nu, \tau) = \alpha$ such that $\alpha_i \leq \alpha_j$ for all $i < j < k$, then $c_A(\xi_j, \xi_i) \leq \alpha_{j-1}$ for all $i < j < k$.

With this information about the sort of witnessing sequences that are available in hand one can extract some more information about inequalities of the type featured in Lemmas (5) and (6). This inequality for decreasing parameter A is completely general (unlike Lemmas (5) and (6) which relied on increasing A only in a very controlled, albeit important, way).

Lemma 11.
$$\forall A,\, B \subseteq \mathcal{F}\, \forall \nu < \tau < \zeta < \kappa^+ \big(A \subseteq B \longrightarrow \\ (c_B(\nu, \tau) \leq c_B(\tau, \zeta) \longrightarrow c_A(\nu, \tau,) \leq c_A(\tau, \zeta)\big).$$

Proof. If $c_A(\tau, \zeta) = \kappa$ there is nothing to be shown as $c_A(\nu, \tau) \leq \kappa$ by definition. If $c_A(\tau, \zeta) = \beta$, say, with $\beta < \kappa$, let $\xi_k = \tau < \xi_{k-1} < \ldots \xi_0 = \zeta$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ be a good witness showing this. Thus $\tau \in \mathrm{rge}(g_{k-1})$.

One has that $c_B(\nu, \tau) \leq c_B(\tau, \zeta) \leq c_A(\tau, \zeta)$ hypothesis and Lemma (11). So by Lemma (4) one has that $\nu \in \mathrm{rge}(g_{k-1})$. Hence $c_A(\tau, \zeta) \leq \beta$, as required, since $\nu < \xi_{k-1} < \ldots \xi_0 = \zeta$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ are good witnesses that $c_A(\nu, \tau) \leq c_A(\tau, \zeta)$.

It is also now possible to prove one of the most important facts about the c_A, that they are subadditive.

Theorem 12. If $A \subseteq \mathcal{F}$ and $\nu < \tau < \zeta < \kappa^+$ then

(i) $$c_A(\nu, \tau) \leq \max(\{\, c_A(\nu, \zeta),\, c_A(\tau, \zeta)\,\}),$$
(ii) $$c_A(\nu, \zeta) \leq \max(\{\, c_A(\nu, \tau),\, c_A(\tau, \zeta)\,\}).$$

Proof. (i) Let $\langle \xi_i \mid i \leq k\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle$ be good witnesses to the value of $c_A(\tau, \zeta) = \alpha$, so that $\langle \alpha_i \mid i < k\rangle$ is strictly increasing, and let $\langle \xi'_i \mid i \leq l\rangle$ and $\langle (\beta_i, g_i) \mid i < l\rangle$ be good witnesses to the value of $c_A(\nu, \zeta) = \beta$ so that $\langle \beta_i \mid i < l\rangle$ is strictly increasing. Note that $\alpha = \alpha_{k-1}$ and $\beta = \beta_{k-1}$.

Suppose first of all that $c_A(\nu, \zeta) > c_A(\tau, \zeta)$. I claim that there is a witnessing sequence to the value of $c_A(\nu, \zeta)$ which features τ as one of the members of the sequence of ordinals. Let $i < l$ be greatest such that $\tau \leq \xi'_j$. There are two cases depending on whether $\beta_j \leq \alpha_{k-1}$ or not.

Suppose $\beta_j \leq \alpha_{k-1}$ and let $j' \in (j, l)$ be least such that $\alpha_{k-1} < \beta_{j'}$. Let $F \in \mathcal{F}_{\alpha\kappa}$ be such that $\zeta \in \mathrm{rge}(F)$. There is some such F because $\zeta \in \mathrm{rge}(f_0)$, by (I.i.ii) in the definition of $(\kappa, 1)$-simplified morasses. (Note, I do not claim that $F \in A$ — in most cases $F \notin A$.) By Lemma (4) one has that τ, $\xi_{k-1} \in \mathrm{rge}(F)$ and so, letting $\bar{\xi}$ be $(\xi_{k-1})_\alpha$, whence $f_{k-1}(\bar{\xi}) = \xi_{k-1}$, one has that $F \restriction \bar{\xi}{+}1 = f_{k-1} \restriction \bar{\xi}{+}1$ and $\mathrm{rge}(F) \cap \xi_{k-1} = \mathrm{rge}(f_{k-1}) \cap \xi_{k-1}$. By Note (5) one has that $c_A(\xi'_{j'}, \zeta) \leq \alpha$, so, by Lemma (4) again, one has that $\xi'_{j'} \in \mathrm{rge}(F)$. As $\xi'_{j'} < \tau$ one has that $\xi'_{j'} \in \mathrm{rge}(f_{k-1})$. Hence $\langle \xi_i \mid i \leq k\rangle^\frown\langle \xi'_i \mid i \in [j', l)\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle^\frown(\alpha, f_{k-1})^\frown\langle (\beta_i, g_i \mid i \in [j', k)\rangle$ are witnesses that $c_A(\nu, \zeta) = \beta$.

Now suppose that $\beta_j > \alpha_{k-1}$. Let $G \in \mathcal{F}_{\beta_j\kappa}$ be such that $\zeta \in \mathrm{rge}(G)$. Note, again, I do not claim that $G \in A$, merely that there is some such G by (I.1.ii) since $\zeta \in \mathrm{rge}(g_0)$. As in the previous argument, Lemma (4) ensures that both ξ'_j and $\tau \in \mathrm{rge}(G)$. So $\tau \in \mathrm{rge}(g_j)$. Consequently $\langle \xi_i \mid i \leq k\rangle^\frown\langle \xi'_i \mid i \in (j, l)\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle^\frown\langle (\beta_i, g_i \mid i \in [j, k)\rangle$ are witnesses that $c_A(\nu, \zeta) = \beta$.

Now it is clear that $c_A(\nu, \tau) = c_A(\nu, \zeta)$ since if $c_A(\nu, \tau) > c(\nu, \zeta)$ then truncating the initial part of the witnessing sequences just produced which include τ in the sequence of ordinals would give a contradiction by Note (5).

Finally, suppose that $c_A(\nu, \zeta) \leq c_A(\tau, \zeta)$. Again, there is some $F \in \mathcal{F}_{\alpha\kappa}$ such that $\tau \in \mathrm{rge}(F)$. By Lemma (4), again, one has that $\tau \in \mathrm{rge}(F)$ and that $\nu \in \mathrm{rge}(F)$. Arguing as in the first case above, one has that $\mathrm{rge}(F) \cap \xi_{k-1} = \mathrm{rge}(f_{k-1}) \cap \xi_{k-1}$. So $\nu \in \mathrm{rge}(f_{k-1})$ and $\langle \tau, \nu\rangle$ and $\langle (\alpha, f_{k-1})\rangle$ are witnesses that $c_A(\nu, \tau) \leq \alpha = c_A(\tau, \zeta)$.

(ii) It is clear that $c_A(\nu, \zeta) \leq \max(\{\, c_A(\nu, \tau), c_A(\tau, \zeta)\,\})$ since if one has witnessing sequences to the values of $c_A(\nu, \tau)$, $c_A(\tau, \zeta)$ then their concatentation is a witness to the inequality.

Observation 13. Let $A \subseteq \mathcal{F}$. If $\alpha < \kappa$ and $\tau < \kappa^+$ then $\overline{\overline{\{\nu \mid c_A(\nu, \tau) \leq \alpha\}}} < \kappa$ (since $\bigcup\{\, \mathrm{rge}(f) \mid (\alpha, f) \in A\}$ is of size less than κ). This gives another useful property shared by c_A and c.

The next lemma is interpolated here because the proof is elementary, although the lemma itself will be useful in arguments for the κ^+-chain condition for various κ-$\mathbb{M}$-proper forcings, (*cf.* §4, [M*4] and [M*5]).

Lemma 14. Suppose $\xi < \zeta < \kappa^+$, $\alpha < \kappa$, $D \subseteq \bigcup\{\mathcal{F}_{\gamma\alpha} \mid \gamma < \alpha\}$ and (α, F), $(\alpha, G) \in \mathcal{F}$. Set $A = \{\, (\gamma, F \cdot g) \mid \gamma < \alpha \;\;\&\;\; g \in D \cap \mathcal{F}_{\gamma\alpha}\}$, $B = \{\, (\gamma, G \cdot g) \mid \gamma < \alpha \;\;\&\;\; g \in D \cap \mathcal{F}_{\gamma\alpha}\}$ and $E = A \cup B$. Suppose $c_E(\xi, \zeta) \leq \alpha$. If $\zeta \in \mathrm{rge}(F)$, then $c_E(\xi, \zeta) = c_A(\xi, \zeta)$, and if $\zeta \in \mathrm{rge}(G)$ then

$c_E(\xi, \zeta) = c_B(\xi, \zeta)$.

Proof. Suppose that $\zeta \in \mathrm{rge}(F)$. Let $c_E(\xi, \zeta) = \beta$ and let $\langle \tau_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ be canonical short witnesses to this. As $\zeta \in \mathrm{rge}(F)$ one has that $\tau_i \in \mathrm{rge}(F)$ by Lemma (4) for each $i < k$. Suppose that $(\gamma_i, g_i) \in B$ for some $i < k$. Then $g_i = G \cdot h_i$ for some $h_i \in \mathcal{F}_{\gamma_i \alpha}$ with $(\alpha_i, h_i) \in A$. But then $\tau_i \in \mathrm{rge}(F) \cap \mathrm{rge}(G)$, so $F((\tau_i)_\alpha) = \tau_i = G((\tau_i)_\alpha)$ and $F((\tau_i)_\alpha + 1) = G \restriction ((\tau_i)_\alpha + 1)$ by Fact (I.3). Consequently one can replace the use of (γ_i, g_i) in the witnessing sequence by $(\gamma_i, F \cdot h_i)$, noting that $(\gamma_i, F \cdot h_i) \in A$ by the definition of A Hence $c_E(\xi, \zeta) \geq c_A(\xi, \zeta)$. So $c_E(\xi, \zeta) = c_A(\xi, \zeta)$ by Lemma (3).

Exactly the same argument with the rôles of A and B exchanged, gives that $c_E(\xi, \zeta) = c_B(\xi, \zeta)$ if $\zeta \in \mathrm{rge}(G)$.

The following definition and lemmas allow one to get a much more concrete grasp on the kind of witnessing sequences to the value of $c_A(\nu, \tau)$ that are available.

Definition 15. Let $A \subseteq \mathcal{F}$, $\tau < \kappa^+$ and $\alpha < \kappa$. Let $\xi_0 = \tau$ and define, by induction, $\xi_{i+1} = \min(\{\mathrm{rge}(g) \cap \xi_i \mid (\gamma, g) \in A \;\; \& \;\; \gamma \leq \alpha \;\; \& \;\; \xi_i \in \mathrm{rge}(g)\})$ and let γ_i to be the least $\gamma \leq \alpha$ such that there is some $(\gamma, g) \in A$ with ξ_i, $\xi_{i+1} \in \mathrm{rge}(g)$. Let k be greatest such that ξ_k is defined. If $\nu < \tau$ and $c_A(\nu, \tau) = \alpha$ then $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ are *canonical short witnesses* to the value of $c_A(\nu, \tau)$ if ξ_i, $\xi_{i+1} \in \mathrm{rge}(g_i)$ for all $i < k$. (See diagram overleaf.)

Clearly the ξ_i and γ_i are uniquely defined and the g_i are only defined uniquely up to agreeing on $(\xi_i)_{\alpha_i} + 1$.

Lemma 16. Let $A \subseteq \mathcal{F}$, $\nu < \tau < \kappa^+$ and $c_A(\nu, \tau) = \alpha < \kappa$. If $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ are canonical short witnesses that $c_A(\nu, \tau) = \alpha$ then $\xi_k = \nu$ and the canonical short witnesses really are witnesses that $c_A(\nu, \tau) = \alpha$.

Proof. It is clear, by induction that $c_A(\xi_i, \tau) \leq \alpha$, by Note (10), and hence that $c_A(\nu, \xi_i) \leq \alpha$ by Theorem (12). As $\langle \xi_i \mid i \leq k \rangle$ is decreasing one has that the definition terminates only when one reaches some $\xi_k = \nu$. Thus canonical short witnesses are witnesses to the value of $c_A(\nu, \tau)$ and $\gamma_{k-1} = \alpha$ as otherwise one would have a contradiction to the minimality of α.

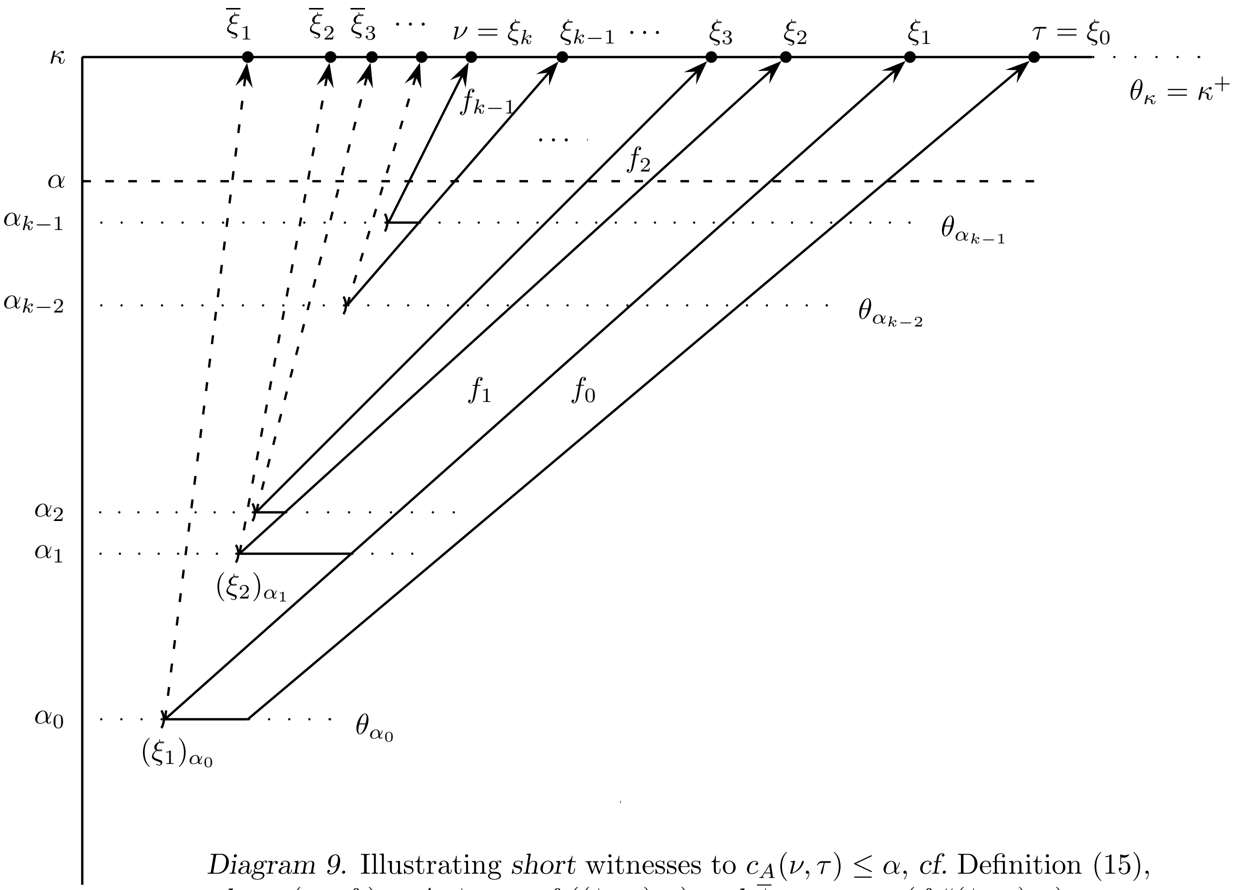

Diagram 9. Illustrating *short* witnesses to $c_A(\nu, \tau) \leq \alpha$, *cf.* Definition (15), where $(\alpha_i, f_i) \in A$, $\xi_{i+1} = f_i((\xi_{i+1})_{\alpha_i})$ and $\overline{\xi}_{i+1} = \text{ssup}(f_i``(\xi_{i+1})_{\alpha_i})$.

The appellation 'canonical short' is justified by the following fact.

Proposition 17. If $A \subseteq \mathcal{F}$, $\nu < \tau < \kappa^+$, $c_A(\nu, \tau) = \alpha < \kappa$, and $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ are canonical short witness to the value $c_A(\nu, \tau)$, then k is the least length of any witnessing sequence showing that $c_A(\nu, \tau) = \alpha$..

Proof. Let $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ be canonical short witnesses to $c_A(\nu, \tau) = \alpha < \kappa$. Suppose that $\langle \zeta_i \mid i \leq l \rangle$ and $\langle (\beta_i, f_i) \mid i < l \rangle$ are witnesses to the value of $c_A(\nu, \tau)$ and of minimal length. Then $\langle \beta_i \mid i < l \rangle$ is strictly increasing or else one could decrease the length of the sequence by refining it as in Lemma (4), contradicting the minimality of l. Moreover suppose that the agreement between initial segments of $\langle \zeta_i \mid i \leq l \rangle$ and $\langle \xi_i \mid i \leq k \rangle$ is of maximal possible length. Suppose this agreement is not of length l. Let i be least such that $\xi_{i+1} \neq \zeta_{i+1}$. Then $\xi_{i+1} < \zeta_{i+1}$ since $\xi_i = \zeta_i$, $\zeta_{i+1} \in \text{rge}(f_i)$ and $f_i \in A$, so $\xi_{i+1} \leq \min(\text{rge}(f_i) \cap \nu) \leq \zeta_{i+1}$ by the definition of ξ_{i+1}. Consequently $\xi_{i+1} \neq \nu$ by the minimality of l. Let j be greatest such that $\xi_{i+1} < \zeta_j$.

If $\gamma_i \leq \beta_j$ then $\xi_{i+1} \in \text{rge}(f_j)$ since there is some $F \in \mathcal{F}_{\beta_j \kappa}$ with $\xi_i \in \text{rge}(F)$ by (I.1.ii). Now $c_A(\zeta_j, \zeta_i) \leq \beta_j$ by Note (10), so $\zeta_j \in \text{rge}(F)$ by Lemma (4). Hence $\text{rge}(F) \cap \zeta_j = \text{rge}(f_j) \cap \zeta_j$ and one can apply Lemma (4) to get that $\xi_{i+1} \in \text{rge}(f_j)$ since $c_A(\xi_{i+1}, \xi) \leq \beta_j$. Now the sequence $\langle \zeta_m \mid m < j \rangle ^\frown \xi_{i+1} {}^\frown \langle \zeta_m \mid m \in (j+1, l) \rangle$ and $\langle (\beta_m, f_m) \mid m < l \rangle$ also witnesses the value of

$c_A(\nu,\tau)$, has minimal length but has a longer agreement of initial segments with the canonical witness sequence than $\langle \zeta_m \mid m < l\rangle$ has, contradicting the maximality of that agreement.

So suppose that $\gamma_i > \beta_j$. Let j' be greatest such that $\gamma_i > \beta_{j'}$. Note that there is some such $j' < l-1$ since $\xi_{i+1} \neq \nu$. Then $c_A(\zeta_{j'+1},\zeta_i) < \gamma_i$ so $\zeta_{j'+1} \in \mathrm{rge}(g_i)$ by Lemma (4) (since $\zeta_i = \xi_i$). Then $\langle \zeta_m \mid m < i\rangle^\frown \xi_{i+1}{}^\frown\langle \zeta_m \mid m \in [j'+1, l)\rangle$ and $\langle (\gamma_m, g_m) \mid m \leq i\rangle^\frown\langle (\beta_m, f_m) \mid m \in [j'+1, l)\rangle$ also witnesses the value of $c_A(\nu,\tau) = \alpha$. This new witness either has length contradicting the minimality of l (if $i \neq j'$) or has has minimal length but has a longer agreement of initial segments with the canonical witness sequence, contradicting the maximality of agreement that was assumed.

Hence one has that $k = l$ and the Proposition is proven.

§2. Canonical long witnesses to connectedness.

Notational reminder. In this section I will make heavy use of the notational conventions recalled in Definition (I.6). In particular, if $\nu < \kappa^+$, $(\beta, f) \in \mathcal{F}$ and $\nu \in \mathrm{rge}(f)$ then $f(\nu_\beta) = \nu$ and ψ^β_ν is the map $f \upharpoonright \nu_\beta + 1$. Both ν_β and ψ^β_ν depend only on β and ν and not on f, by Fact (I.3). Sometimes I shall need to apply this notation to an ordinal signified by a Greek letter with a subscript, such as ξ_i. If $\xi_i \in \mathrm{rge}(f)$ for some $(\beta, f) \in \mathcal{F}$ I then write $(\xi_i)_\beta$ for the ordinal $\overline{\xi}$ such that $f(\overline{\xi}) = \xi_i$.

Definition 1. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. Set

$$\begin{aligned}
&A_\alpha(\nu) = \{\xi < \nu \mid \exists(\gamma, g) \in A\ (\gamma \leq \alpha\ \ \&\ \ \nu, \xi \in \mathrm{rge}(g))\},\\
&\Gamma(\nu) = \{\gamma \leq \alpha \mid \exists(\gamma, g) \in A\ \ \nu \in \mathrm{rge}(g)\},\\
&\gamma^* = \gamma^*(\nu) = \sup(\Gamma(\nu)) \text{ and}\\
&\psi_\nu = \psi^{\gamma^*(\nu)}_\nu.
\end{aligned}$$

So $A_\alpha(\nu)$ is the collection of ordinals ξ less than ν such that $c(\xi,\nu) \leq \alpha$ and this is demonstrated by some map in A. Obviously $\Gamma(\nu)$, γ^* and ψ_ν depend on A and α as well as ν, but as A and α will be fixed in the discussion in this section and the next the notation suppresses this dependence.

Lemma 2. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. Then

$$A_\alpha(\nu) = \bigcup \big\{\, \psi^\gamma_\nu “\nu_\gamma \mid \exists(\gamma, g) \in A\ \big(\beta \leq \alpha\ \ \&\ \ \nu, \xi \in \mathrm{rge}(g)\big) \,\big\}.$$

Proof. Immediate from the definition of the ψ^γ_ν.

Then next proposition shows that each $A_\alpha(\nu)$ is the range of a single map derived from $\mathbb{M}$, albeit that this map need not be (a truncation of) a map in A.

Proposition 3. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. Then $A_\alpha(\nu) = \psi_\nu “\nu_{\gamma^*}$.

Proof. It is clear that $A_\alpha(\nu) \subseteq \psi_\nu “\nu_{\gamma^*}$. For if $\nu \in \mathrm{rge}(g)$, $(\gamma, g) \in A$ and $\gamma \leq \alpha$ then $\gamma \leq \gamma^*$, by the definition of γ^*, and there are some $(\gamma^*, g^*) \in \mathcal{F}$ and $h \in \mathcal{F}_{\gamma\gamma^*}$ such that $g = g^* \cdot h$, by Definition (I.1.ii). But then $\nu = g(\nu_\gamma) = g^*(h(\nu_\gamma))$ and $h(\nu_\gamma) = \nu_{\gamma^*}$, and $h“\nu_\gamma \subseteq h(\nu_\gamma)$, so $g“\nu_\gamma \subseteq g^*“\nu_{\gamma^*}$.

The converse inequality is clear by Lemma (2) if γ^* is the maximal element of $\Gamma(\nu)$. On the other hand, if $\Gamma(\nu)$ has no maximal element then γ^* is a limit ordinal and if $\xi \in \psi_\nu``\nu_{\gamma^*}$ there is some $\gamma < \gamma^*$ and some map $h \in \mathcal{F}_{\gamma\gamma^*}$ such that $\xi_{\gamma^*}$, $\nu_{\gamma^*} \in \mathrm{rge}(h)$, by Definition (I.1.iv). Thus $\xi_{\gamma^*} \in \psi_{\langle\gamma,\nu_\gamma\rangle,\langle\gamma^*,\nu_{\gamma^*}\rangle}``\nu_\gamma$. So for any $\gamma' \in \Gamma(\nu)\setminus\gamma$ one has $\xi_{\gamma^*} \in \psi_{\langle\gamma',\nu_{\gamma'}\rangle,\langle\gamma^*,\nu_{\gamma^*}\rangle}``\nu_{\gamma'}$ as well (by the same argument as in the previous paragraph using (I.i.ii)). Hence $\xi \in \psi_\nu \cdot \psi_{\langle\gamma',\nu_{\gamma'}\rangle,\langle\gamma^*,\nu_{\gamma^*}\rangle}``\nu_{\gamma'} = \psi_\nu^{\gamma'}``\nu_{\gamma'}$, while $\psi_\nu^{\gamma'}``\nu_{\gamma'} \subseteq A_\alpha(\tau)$, by Lemma (2).

Definition 4. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. In view of Lemma (3), say that ψ_ν *instantiates* $A_\alpha(\nu)$. Occasionally I say that (γ, g), or ψ_τ^γ, *instantiates* $A_\alpha(\nu)$ if $\gamma = \gamma^*(\nu)$ and $g \restriction \nu_\gamma + 1 = \psi_\nu$, or $\psi_\tau^\gamma \restriction \nu_\gamma + 1 = \psi_\nu$, respectively,

Note it may be that there is no map $(\gamma^*, g) \in A$ such that $\nu \in \mathrm{rge}(g)$. However if there is some such (γ^*, g) then at times it will also be said that this map instantiates $A_\alpha(\nu)$ since $\psi_\nu \restriction \nu+1 = g \restriction \nu+1$.

Lemma 5. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. If $\nu < \kappa^+$, $(\beta, f) \in A$, $\beta \le \alpha$ and $\nu \in \mathrm{rge}(f)$ then $\mathrm{rge}(f) \cap \nu \subseteq A_\alpha(\nu)$.

Proof. Immediate from the definition of $A_\alpha(\nu)$.

Corollary 6. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. If $\nu < \tau < \kappa^+$ and $\nu \in A_\alpha(\tau)$ then $A_\alpha(\tau) \cap \nu \subseteq A_\alpha(\nu)$. Equality holds if $\gamma^*(\nu) = \gamma^*(\tau)$.

Proof. Let $(\gamma, g) \in A$ be such that $\gamma \le \alpha$ and τ, $\nu \in \mathrm{rge}(g)$. Then for all $\gamma' \in [\gamma, \gamma^*]$ one has that there are $f \in \mathcal{F}_{\gamma\gamma'}$ and $(\gamma', h) \in \mathcal{F}$ such that $g = h \cdot f$, and if $(\gamma', g') \in A$ and $\tau \in \mathrm{rge}(g)$ then $g' \restriction \tau_{\gamma'} = h \restriction \tau_{\gamma'}$, and hence $\nu \in \mathrm{rge}(g')$. Thus if $\xi \in A_\alpha(\tau) \cap \nu$ then there is some $(\gamma, g) \in A$ with ν, τ and $\xi \in \mathrm{rge}(g)$. So $A_\alpha(\tau) \cap \nu \subseteq A_\alpha(\nu)$.

Suppose $\gamma^*(\nu) = \gamma^*(\tau)$, $= \gamma^*$, say. One has $A_\alpha(\tau) \cap \nu = (\psi_\tau^{\gamma^*}``\tau_{\gamma^*}) \cap \nu$ and $A_\alpha(\nu) = \psi_\nu^{\gamma^*}``\nu_{\gamma^*}$. But $\nu \in \mathrm{rge}\,\psi_\tau^{\gamma^*}$, so $\psi_\tau^{\gamma^*} \restriction \nu_{\gamma^*} = \psi_\nu^{\gamma^*} \restriction \nu_{\gamma^*}$, by Fact (I.4), giving that $A_\alpha(\nu) = A_\alpha(\tau) \cap \nu$.

The next proposition is similar in spirit to the definition of canonical short witnessing sequences, and one could define canonical short witnessing sequences through the inductive definition outlined in the proposition.

Proposition 7. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$, and $\nu < \tau < \kappa^+$. Define a sequence of ordinals ξ_i inductively. Set $\xi_0 = \tau$ and let $\xi_{i+1} = \min(A_\alpha(\xi_i) \setminus \nu)$ if $A_\alpha(\xi_i) \setminus \nu$ is non-empty, and stop the induction otherwise. As $\xi_{i+1} < \xi_i$ if both are defined the induction clearly stops after finitely many steps by the well-foundedness of the ordinals. Let $k - 1$ be the greatest i such that $\nu < \xi_i$, so that either $\nu \in A_\alpha(\xi_{k-1})$ and $\xi_k = \nu$ or $A_\alpha(\xi_{k-1}) \setminus \nu = \emptyset$ and ξ_k is undefined. (Observe that if ξ_{i+1} is defined then there is some $(\gamma, g) \in A$ with $\gamma \le \gamma^*(\xi_i)$ and ξ_{i+1}, $\xi_i \in \mathrm{rge}(g)$.)

If $c_A(\nu, \tau) \le \alpha$, then (a) $c_A(\nu, \xi_i)$, $c_A(\xi_i, \tau) \le \alpha$, and (b) $A_\alpha(\xi_i)\setminus\nu \ne \emptyset$, for all $i < k$. Consequently, $\xi_k = \nu$.

One also has that $\langle \gamma^*(\xi_i) \mid i < k\rangle$ is strictly increasing. And that if $\langle (\gamma_i, g_i) \mid i < k\rangle$ is such that $(\gamma_i, g_i) \in A$ and ξ_{i+1}, $\xi_i \in \mathrm{rge}(g_i)$ then $\gamma_i \le \gamma^*(\xi_i) < \gamma_j$ for $i < j < k$ and $\langle \xi_i \mid i \le k\rangle$ and $\langle (\gamma_i, g_i) \mid i < k\rangle$ are good witnesses to $c_A(\nu, \tau) \le \alpha$. They are canonical short witnesses to the value of $c_A(\nu, \tau)$ if $\gamma^*(\xi_{k-1}) \le c_A(\nu, \tau)$ and γ_i is the minimal γ such that there is some $(\gamma, g) \in A$

with ξ_{i+1}, $\xi_i \in \mathrm{rge}(g)$ for each $i < k$. Finally, k is the miminal length of a witnessing sequence to $c_A(\nu, \tau) \leq \alpha$.

Proof. The proof of (a) and (b) is by an interwoven induction on $k-1$. It is trivial that (a) holds for $i = 0$. Suppose one has that (a) for i. If (β, f) is the first element of A in any sequence showing that $c_A(\nu, \xi_i) \leq \alpha$ then $\mathrm{rge}(f) \cap [\nu, \xi_i) \neq \emptyset$ and so $A_\alpha(\xi_i) \setminus \nu \neq \emptyset$ as $\mathrm{rge}(f) \cap \nu \subseteq A_\alpha(\xi_i)$ by Lemma (5), so (b) holds for i. On the other hand, if (b) holds for i then ξ_{i+1} is defined and $\xi_{i+1} < \xi_i$. As $c_A(\xi_i, \tau) \leq \alpha$ and $c_A(\xi_{i+1}, \xi_i) \leq \alpha$ one has that (a) holds for $i+1$ by the subadditivity of c_A.

As noted in the definition of the ξ_i in the statement of the proposition, the consequence of (b) for $i = k-1$ is that $\xi_k = \nu$.

$\langle \gamma_i \mid i < k \rangle$ is a strictly increasing sequence by Lemma (6). If $\langle (\gamma_i, g_i) \mid i < k \rangle$ is as in the statement of the Lemma it is clearly a witnessing sequence that $c_A(\nu, \tau) \leq \alpha$, and such sequences do exist by the observation in the statement of the Lemma. Also $\gamma^*(\xi_i) < \gamma_{i+1}$ if $i + 1 < k$ since otherwise one has a contradiction to the minimality of ξ_{i+1} in $A_\alpha(\xi_i) \setminus \nu$. Thus these witnesses are good. They are clearly canonical short witnesses if $\gamma^*(\xi_{k-1}) \leq c_A(\nu, \tau)$. Finally k is the minimal length of any witness sequence that $c_A(\nu, \tau) \leq \alpha$ by an argument almost identical to Proposition (1.10). I leave the details to the reader.

Definition 8. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$, and $\nu < \tau < \kappa^+$. As in the statement of Proposition (7), let $\xi_0 = \tau$ and inductively set $\xi_{i+1} = \min(A_\alpha(\xi_i) \setminus \nu)$ if $A_\alpha(\xi_i) \setminus \nu$ is non-empty, stopping the induction otherwise. Suppose that $c_A(\nu, \tau) \leq \alpha$. Then $\langle \xi_i \mid i \leq k \rangle$ and $\langle \psi_{\xi_i} \mid i < k \rangle$ are *canonical short high witnesses* that $c_A(\nu, \tau) \leq \alpha$.

The idea behind canonical short high witnesses is much less complicated than the portmanteau Proposition (7) may make it appear and, in fact, is quite simple. One would like to start at τ and find a map in A which starts as far up the morass below $\alpha + 1$ as possible with τ in its range. One would then go as far along the range of this map towards ν as possible and make a note of the ordinal one arrives at. One would repeat the process, bounding along towards ν as fast as one could until one actually arrived at ν in this way, which one would do, with no possibility of overstepping it, provided that $c_A(\nu, \alpha) \leq \alpha$. However, if A is not finite, one cannot necessarily find maps in A with maximal heights below $\alpha + 1$ with particular ordinals in their range, so one has to do the next best thing and use the morass maps ψ_{ξ_i} derived from A instead. Making use of these maps instead, one repeatedly travels along their ranges until one finds ν. (Of course if A is finite then one actually proceeds in the fashion desired at the start of the paragraph.)

Lemma (7) says that this is the fastest way to get from τ to ν using maps from A which are from levels at or below α, in the sense of having the fewest intermediate ordinals to be noted down and ranges of maps to be traced along, *i.e.* the lemma shows canonical short high witnesses are the shortest possible sequences witnessing $c(\nu, \tau) \leq \alpha$.

But it turns out that canonical short (high) witnesses are by no means the only useful canonical (good) way of journeying from τ to(wards) ν. The next part of this section discusses canonical *long* witnesses. The idea here is essentially that one takes steps that are as short as is reasonable. Instead of going as far along each successive map towards ν as possible, one only goes as far towards ν as one can while it is still true that the map instantiates $A_\alpha(\eta)$ for the η that one is passing through.

Then one has to stumble about a little to find the next map to journey along. Fortunately the same map is fine for a final segment of the ordinals where one might chance to start the next step, so one really can journey along the length of the range of this map, until it too no longer instantiates the $A_\alpha(\eta)$ and one changes map again. Eventually one does finally reach ν if $c_A(\nu,\tau) \leq \alpha$. It is reasonably clear that this procedure takes (in general) many more steps than canonical short high witnesses, and as the definition can plausibly be described as canonical the name of canonical long witnesses is perhaps reasonable. In §3, when one sees the rôle played by the canonical long witnesses in the analysis of Koszmider's local distance function from [K00], justification for calling this sequence of maps and ordinals *the* canonical long witnesses will become more evident, because while one can increase the length of the sequence witnessing $c_A(\nu,\tau) \leq \alpha$ one cannot in this way increase the distance which will be derived from the witnessing sequence.

Definition 9. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. Let ψ_ν instantiate $A_\alpha(\nu)$. Let

$$C_\alpha(\nu) = \{\, \xi \in A_\alpha(\nu) \mid \psi_\nu \upharpoonright \xi_{\gamma^*(\nu)} \text{ instantiates } A_\alpha(\xi) \,\}.$$

By Corollary (6) one has that $C_\alpha(\nu) = \{\, \xi \in A_\alpha(\nu) \mid \gamma^*(\nu) = \gamma^*(\xi) \,\}$.

Lemma 10. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. If $\xi < \zeta$ and ξ, $\zeta \in A_\alpha(\nu)$ then there is some $(\beta, f) \in A$ such that ξ, ζ, and $\nu \in \mathrm{rge}(f)$.

Proof. If $\Gamma(\nu)$ has a maximal element then any pair $(\gamma^*(\nu), h)$ with $\nu \in \mathrm{rge}(h)$ also has ξ, $\zeta \in \mathrm{rge}(h)$. If $\Gamma(\nu)$ has no maximal element then $\gamma^*(\nu)$ is a limit ordinal and there is some $(\varepsilon, h) \in A$ with ξ, ζ, $\nu \in \mathrm{rge}(h)$ by Definition (I.1.ii).

Lemma 11. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. If $\xi < \zeta$ and ξ, $\zeta \in A_\alpha(\nu)$ then $\gamma^*(\zeta) \leq \gamma^*(\xi)$.

Proof. By Lemma (10), fix some $(\beta, f) \in A$ such that ξ, ζ, and $\nu \in \mathrm{rge}(f)$. If one had that $\gamma^*(\zeta) > \gamma^*(\xi)$ then there is some $(\gamma, g) \in A$ such that $\gamma^*(\xi) < \gamma \leq \gamma^*(\zeta)$ and $\zeta \in \mathrm{rge}(g)$. Factor f as $f = f'' \cdot f'$ where $(\gamma, f'') \in \mathcal{F}$ and $f' \in \mathcal{F}_{\beta\gamma}$. Then $\zeta \in \mathrm{rge}(f'') \cap \mathrm{rge}(g)$, so $f'' \upharpoonright \zeta_\gamma = g \upharpoonright \zeta_\gamma$ and, hence, $\xi \in \mathrm{rge}(g)$. This contradicts the supposition that $\gamma^*(\xi) < \gamma$.

Lemma 12. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. Then $C_\alpha(\nu)$ is a final segment of $A_\alpha(\nu)$ for each $\nu < \kappa^+$.

Proof. If $C_\alpha(\nu) = \emptyset$ the lemma is trivially true. Otherwise, let $\xi \in C_\alpha(\nu)$ and let $\zeta \in A_\alpha(\nu) \setminus \xi + 1$. By Lemma (11) $\gamma^*(\xi) \geq \gamma^*(\zeta)$. As $\gamma^*(\nu) = \gamma^*(\xi)$ this gives that $\gamma^*(\nu) = \gamma^*(\zeta)$

Corollary 13. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. If ξ, $\zeta \in C_\alpha(\nu)$ and $\xi < \zeta$ then $\xi \in C_\alpha(\zeta)$ and $A_\alpha(\zeta) \setminus \xi = A_\alpha(\nu) \cap [\xi, \zeta)$.

Note 14. Even if $\xi \in A_\alpha(\nu)$, $A_\alpha(\xi) = A_\alpha(\nu) \cap \xi$ and $\xi < \zeta \in A_\alpha(\nu)$ it may be the case that $A_\alpha(\zeta) \neq A_\alpha(\nu) \cap \zeta$. For it could be, for example, that (β, f) instantiates $A_\alpha(\nu)$, but that there is some $(\gamma, g) \in A$ with $\beta < \gamma$ and $\zeta \in \mathrm{rge}(g)$ and, letting $f = k \cdot h$ where $k \in \mathcal{F}_{\gamma\kappa}$ and $h \in \mathcal{F}_{\beta\gamma}$, one has that $h \upharpoonright (\xi_\beta + 1) = \mathrm{id} \upharpoonright (\xi_\beta + 1)$ and $h(\nu_\beta) \neq \nu_\beta$. Thus if $c_A(\nu, \tau) \leq \alpha$ and $\eta \in A_\alpha(\xi_i) \setminus \nu$, whence $c_A(\nu, \tau) \leq \alpha$, it may be that the canonical short high witnesses to the latter are not just a truncation of the canonical short high witnesses to the former.

Lemma 15. Suppose $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. There is some $\eta < \nu$ and some $\beta \leq \alpha$ such that ψ^β_τ instantiates $A_\alpha(\tau)$ for all $\tau \in A_\alpha(\nu) \setminus \eta$. (Note that for all $\tau \in A_\alpha(\nu) \setminus \eta$ one has that $\psi^\beta_\tau = \psi^\beta_\nu \restriction \tau + 1$.)

Proof. If $A_\alpha(\nu)$ has a maximal element then the lemma is trivial. If not, then by Lemma (11) $\langle \gamma^*(\tau) \mid \tau \in A_\alpha(\nu) \rangle$ is a decreasing sequence of ordinals, so attains its minimum at some $\eta \in A_\alpha(\nu)$. By Lemma (11), ψ^β_τ instantiates $A_\alpha(\tau)$ for all $\tau \in A_\alpha(\nu) \setminus \eta$.

Definition 16. Suppose that $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. Let $\eta < \nu$ and $\beta \leq \alpha$ be as given by Lemma (15), so that ψ^β_τ instantiates $A_\alpha(\tau)$ for all $\tau \in A_\alpha(\nu) \setminus \eta$. (Recall, again, that $\psi^\beta_\tau = \psi^\beta_\nu \restriction \tau_\beta$ for all $\tau \in A_\alpha(\nu)$.) Set

$$\psi^*_\nu = \psi^\beta_\nu \quad \text{and} \quad C'_\alpha(\nu) = \{\xi \in A_\alpha(\nu) \mid \psi^*_\nu \text{ instantiates } A_\alpha(\xi)\}.$$

Lemma 17. Suppose $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. If $\xi \in C'_\alpha(\nu)$ and $\rho \in A_\alpha(\nu) \setminus \xi$, then $\rho \in C'_\alpha(\nu)$.

Proof. Immediate from Lemma (11).

Lemma 18. Suppose $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \kappa^+$. If $C_\alpha(\nu) \neq \emptyset$ then $C'_\alpha(\nu) = C_\alpha(\nu)$.

Proof. Immediate from the definitions.

Lemma 19. If $A \subseteq \mathcal{F}$, $\alpha < \kappa$, $\nu < \kappa^+$ and $\rho \in C'_\alpha(\nu)$, then $C_\alpha(\rho) \cap A_\alpha(\nu) = C'_\alpha(\nu) \cap \rho$.

Proof. Again immediate from the definitions.

Definition 20. Let $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. Suppose $\nu < \tau < \kappa^+$. Set $\zeta_0 = \tau$ and make the following inductive definition. If $\nu \in C'_\alpha(\zeta_i)$ then let $\zeta_{i+1} = \nu$. If $\nu \notin C'_\alpha(\zeta_i)$ let $\zeta_{i+1} = \min(C'_\alpha(\zeta_i))$. (So the induction continues as long as ν is not found in one of the sets mentioned in the definition and stops, without the next ζ being defined, when ν is found or one of the sets is empty.) Let $\psi^*_{\zeta_i}$ instantiate $A_\alpha(\eta)$ for $\eta \in C'_\alpha(\zeta_i)$ be as given in the definition of $C'_\alpha(\zeta_i)$. (Note that $\gamma^*(\zeta_i) < \gamma_i$, so that $\psi^*_{\zeta_i} \neq \psi_{\zeta_i}$, for $1 \leq i < l$.)

Theorem 21. The sequence of 'ζ_i's defined in Definition (20) is strictly decreasing and so has finite length, say l. If $c_A(\nu, \tau) \leq \alpha$ then $\nu \in C'_\alpha(\zeta_{l-1})$.

Proof. It is clear since each of the sets $C_\alpha(\zeta_i)$, $C'_\alpha(\zeta_i) \subseteq \zeta_i$ that $\zeta_{i+1} < \zeta_i$ whenever ζ_{i+1} is defined, so the inductive definition does stop.

If $\nu \in C_\alpha(\tau)$ then $\zeta_1 = \nu$ and there is nothing more to prove. So suppose that $\nu \notin C_\alpha(\tau)$. The remainder of the proof of the theorem is broken into a series of lemmas (Lemmas (22) to Corollary (27)).

Lemma 22. If $\eta \in C_\alpha(\tau) \cup \{\tau\}$ then $c_A(\eta, \tau) \leq \alpha$ and $c_A(\nu, \eta) \leq \alpha$.

Proof. As $c_A(\nu, \tau) \leq \alpha$, by taking the first map in any sequence witnessing this, it is clear that there is some $(\beta, f) \in A$ such that $\beta \leq \alpha$ and $\tau \in \text{rge}(f)$. So $\gamma^*(\tau)$ is defined, and is at most α. Now, by Lemma (10), for any $\rho \in A_\alpha(\tau)$ one has that there is some $(\gamma, g) \in A$ such that ρ, $\tau \in \text{rge}(g)$. Then $\gamma \leq \gamma^*(\tau) \leq \alpha$. So one has that $c_A(\eta, \tau) \leq \alpha$ and, by Theorem (1.12), the subadditivity of c_A, that $c_A(\nu, \eta) \leq \alpha$.

Lemma 23. If $\eta \in C'_\alpha(\zeta_i) \setminus \nu$ then $c_A(\nu, \eta)$, $c_A(\eta, \tau) \leq \alpha$.

Proof. By induction on i. Suppose one has that $\nu \leq \zeta_i$ and that $c_A(\nu, \zeta_i) \leq \alpha$. (For $i = 0$ this is true by the hypothesis of the theorem.) If $\eta \in C'_\alpha(\zeta_i) \setminus \nu \subseteq A_\alpha(\zeta_i)$ then, by Lemma (10), clearly $c_A(\eta, \zeta_i) \leq \gamma^*(\zeta_i) \leq \alpha$. And, by the subadditivity of c_A, Theorem (1.12), again, $c_A(\nu, \eta)$, $c_A(\eta, \tau) \leq \alpha$. As $\zeta_{i+1} = \min(C'_\alpha(\zeta_i))$ if it is defined, the induction hypothesis is maintained if $\nu \leq \zeta_{i+1}$.

Lemma 24. If ζ_i is defined and $\nu < \zeta_i$ one has that $A_\alpha(\zeta_i) \setminus \nu \neq \emptyset$ and so $C'_\alpha(\zeta_i) \setminus \nu \neq \emptyset$.

Proof. As $c_A(\nu, \zeta_i) \leq \alpha$ by Lemma (23), let (β, f) be the first map in any sequence witnessing this. One has that $\zeta_i \in \mathrm{rge}(f)$ and ζ_i is not minimal in $\mathrm{rge}(f) \setminus \nu$. Since $\mathrm{rge}(f) \cap [\nu, \zeta_i) \subseteq (\psi_{\zeta_i}``(\zeta_i)_{\gamma^*(\zeta_i)} \setminus \nu) = A_\alpha(\zeta_i) \setminus \nu$ one has that $A_\alpha(\zeta_i) \setminus \nu \neq \emptyset$. As, by definition, $C'_\alpha(\zeta_i)$ is a final segment of $A_\alpha(\zeta_i)$, the second half of the conclusion is immediate.

Corollary 25. If $\nu < \zeta_i$ then ζ_{i+1} can only be undefined if $\nu \in C'_\alpha(\zeta_i)$ or $\min(C'_\alpha(\zeta_i) < \nu$.

Lemma 26. Suppose $\nu < \zeta_i$ and $\min(C'_\alpha(\zeta_i)) \leq \nu$. Then $\nu \in C'_\alpha(\zeta_i)$.

Proof. If $\nu = \min(C'_\alpha(\zeta_i))$ there is nothing to prove. So suppose otherwise. One has that $\psi^*_{\zeta_i} \restriction \eta_{\gamma^*(\zeta_i)} + 1$ instantiates $A_\alpha(\eta)$ for every $\eta \in C'_\alpha(\zeta_i)$. As $C'_\alpha(\zeta_i)$ is a final segment of $A_\alpha(\zeta_i)$ there is no $\eta \in A_\alpha(\zeta_i) \setminus \nu$ such that there is some $(\beta, f) \in A$ with $\eta \in \mathrm{rge}(f)$ and $\beta > \gamma_i$. But $c_A(\nu, \zeta_i) \leq \alpha$, so, by applying Lemma (1.4), it must be that $\nu \in \mathrm{rge}(\psi^*_{\zeta_i})$. Hence $\nu \in C'_\alpha(\zeta_i)$ by Lemma (11).

Corollary 27. The definition of the ζ_i stops at some finite stage, say, l, and $\nu \in C'_{\zeta_{l-1}}$, whence $\zeta_l = \nu$, thus completing the proof of Theorem (21).

Lemma 28. Let $A \subseteq \mathcal{F}$, $\alpha < \kappa$, and $\nu < \tau < \kappa^+$. Let $\langle \zeta_i \mid i < l \rangle$ be as defined in Definition (20). Suppose that $c_A(\nu, \tau) \leq \alpha$. Let $\eta \in C'_\alpha(\zeta_i) \setminus \nu$ (or $\eta = \tau$ if $C_\alpha(\tau) \neq \emptyset$). Then $A_\alpha(\eta) \cap C'_\alpha(\zeta_{i+1}) \neq \emptyset$ if $i < l - 1$.

Proof. Immediate as $A_\alpha(\eta) \cap \zeta_{i+1} = A_{\zeta_{i+1}}$ and $C'_\alpha(\zeta_{i+1})$ is a non-empty final segment of $A_{\zeta_{i+1}}$ if $i < l - 1$.

Definition 29. Suppose $c_A(\nu, \tau) \leq \alpha$. The sequences $\nu = \zeta_l < \ldots < \zeta_0 = \tau$ and $\psi^*_{\zeta_i}$ for $i < l$ given by Definition (20) are *canonical long witnesses* that this is so. Note that Theorem (21) proves that one could re-write the definition of the ζ_i as $\zeta_0 = \tau$ and $\zeta_{i+1} = \min(C'_\alpha(\zeta_i) \setminus \nu)$. (*cf.* Diagram (10) after Theorem (3.7).)

Canonical long witnesses are not witnesses in the sense that they directly show that $c_A(\nu, \tau) \leq \alpha$. As well as it being possible that the maps $\psi^*_{\zeta_i}$ are not restrictions of maps in A, one also has that although $\zeta_{i+1} \in \mathrm{rge}(\psi^*_{\zeta_i})$ nevertheless $\zeta_i \notin \mathrm{rge}(\psi^*_{\zeta_i})$ for $0 < i < l$ (and for $i = 0$ as well if $C_\alpha(\tau) = \emptyset$). Rather they are a sort of limit of the possible witnesses. By Lemma (28) one can pick ζ'_{i+1} for $i < l - 1$ with $\zeta'_{i+1} \in A_\alpha(\zeta_i)$ and $A_\alpha(\zeta'_{i+1}) = A_\alpha(\zeta_{i+1})$ which together with the maps $\psi_{\zeta'_i} (= \psi^*_{\zeta_i} \restriction (\zeta'_i)_{\gamma^*(\zeta_i)} + 1)$ are witnesses in the sense that canonical short high witnesses are (making the convention of setting $\zeta'_0 = \zeta_0 = \tau$). That is one can replace each of the witnessing maps by maps which genuinely are in A which do witness that $c(\zeta'_{i+1}, \zeta_i) \leq \gamma^*(\zeta_i)$. However one cannot find witnesses higher up the morass, or, in a sense discussed in the following section, slower.

Canonical long witnesses are different from canonical short high witnesses in an important way in that if $c_A(\nu,\tau)\le\alpha$ and $\langle\zeta_i \mid i\le l\rangle$ is the sequence of ordinals featured in canonical long witnesses and $\eta\in C'_\alpha(\zeta_i)\backslash\nu$ then a truncation (*i.e.* final segment) of a canonical long witness that $c_A(\nu,\tau)\le\alpha$ is a canonical long witness that $c_A(\nu,\eta)\le\alpha$. (By truncation, I mean truncation up to changing the initial ordinal if η is not one of the ζ_i.) Although this observation is immediate from Definition (20) it is sufficiently important to be glorified with the title of a lemma.

Lemma 30. Suppose that $c_A(\nu,\tau)\le\alpha$, $\langle\zeta_i \mid i\le l\rangle$ and $\psi^*_{\zeta_i}$ for $i<l$ are canonical long witnesses to this and $\eta\in C'_\alpha(\zeta_{i^*})\setminus\nu$ for some $i^*<l$. Then $\langle\zeta_i \mid i\le i^*\rangle^\frown\eta$ and $\psi^*_{\zeta_i}$ for $i\le i^*$ are canonical long witnesses that $c_A(\eta,\tau)\le\alpha$ and $\eta^\frown\langle\zeta_i \mid i^*<i\rangle$ and $\psi^*_{\zeta_i}$ for $i^*<i$ are canonical long witnesses that $c_A(\nu,\eta)\le\alpha$.

Proof. Immediate from Definition (20). (This is the point of making Definition (20) the way is done there (and proving Theorem (21)) rather than using the equivalent as proved by Theorem (20) that $\zeta_{i+1}=\min(C'_\alpha(\zeta_i)\setminus\nu)$.)

Lemma 31. Suppose $c_A(\nu,\tau)\le\alpha$. Then $A_\alpha(\tau)\setminus\nu\subseteq\bigcup\{C'_\alpha(\zeta_i)\setminus\nu \mid \nu<\zeta_i\}$.

Proof. Let $\eta\in A_\tau\setminus\nu$. Then there is some $i<l-1$ such that $\zeta_{i+1}\le\eta<\zeta_i$. For all $j<l$ one has that $\zeta_{j+1}\in A_\alpha(\zeta_j)$, so, by repeatedly applying Corollary (6), $\eta\in A_\alpha(\zeta_i)$. As $C'_\alpha(\zeta_i)\cap[\zeta_{i+1},\zeta_i)=A_\alpha(\zeta_i)\cap[\zeta_{i+1},\zeta_i)$, one has that $\eta\in C'_\alpha(\zeta_i)$ for some $i<l$.

Corollary 32. Suppose $c_A(\nu,\tau)\le\alpha$ and $\langle\zeta_i \mid i\le l\rangle$ is the sequence of ordinals from the canonical long witness for this. Suppose $\eta\in C'_\alpha(\zeta_i)$ for $i<l$. Then

$$A_\alpha(\eta)\setminus\nu\subseteq\bigcup\{C'_\alpha(\zeta_j)\setminus\nu \mid \nu<\zeta_j \ \& \ \zeta_j<\eta\}.$$

Proof. Immediate from Lemma (30) and Lemma (31).

This concludes the current consideration of the canonical long witnesses to $c_A(\nu,\tau)\le\alpha$ for some $\nu<\tau<\kappa^+$. There are other interesting canonical witnessing sequences which can be defined. One example, just to demonstrate the possibilities, is the following.

Definition 33. Let $A\subseteq\mathcal{F}$ and $\alpha<\kappa$. Let $B_\alpha(\nu)=\mathrm{rge}(f)\cap\nu$ where $(\beta,f)\in A$, $\nu\in\mathrm{rge}(f)$ and β is *minimal* in α showing this.

Definition 34. Let $A\subseteq\mathcal{F}$ and $\alpha<\kappa$. Let $\nu<\kappa^+$ and $(\beta,f)\in A$ be such that $\nu\in\mathrm{rge}(f)$ and β is *minimal* in α showing this. Let $(\gamma,g)\in A$ be such that γ is minimal greater than β such that $\nu\in\mathrm{rge}(g)$. Let

$$B'_\alpha(\nu)=\{\xi\in A_\alpha(\nu) \mid (\gamma,g)\in A \text{ is minimal such that } \xi\in\mathrm{rge}(g)\}.$$

Definition 35. Suppose that $c_A(\nu,\tau)\le\alpha$. Let $\xi_0=\tau$, $\xi_1=\min(B_\alpha(\tau))$, and $\xi_{i+1}=\min(B'_\alpha(\xi_i))$ for $i>0$, and let $(\gamma_i,g_i)\in A$ be as in the definition of $B'_\alpha(\xi_i)$.

Corollary 35. (to proof of Proposition (7).) One could insist that $\xi_{i+1}\in B'_\alpha(\xi_i)$ for each $i<k$ in the conclusion of Proposition (7) instead of $\xi_{i+1}\in A_\alpha(\xi_i)$.

The procedure of Definition (34) amounts to going as far down into the simplified morass as possible to find witnessing maps rather than staying as close to the surface as possible as is the case with canonical short high witnesses. Where canonical long witnesses give rise to 'long' distances (see the following section), these witnesses seem to give rise to 'shortest' distances.

§3. Local distance functions.

In this section I show how the analysis of the function c_A and the canonical witnesses in the previous section helps give a more explicit insight into the local distance function defined in [K00].

The natural (global) distance function associated with the coupling function c of Definition (I.7) is the function D given in Definition (I.9) by $D(\nu,\tau) = \mathrm{otp}(\tau_{c(\nu,\tau)} \setminus \nu_{c(\nu,\tau)})$. There are a number of ways of defining local distance functions analogous to D in the sense that c_A is an analogue of c. For example one can measure the tracing along the ranges of the maps of a canonical short high witness or along the analogous sequences given by the $B'_\alpha(\eta)$. Koszmider took a seemingly different approach.

Definition 1. ([K00, Definition 2.11] for $\kappa = \omega_1$ and A finite.) Suppose $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. For $\nu \leq \tau < \kappa^+$ set

$$\begin{aligned} d_{A,\alpha}(\nu,\tau) &= \sup(\{\, d_{A,\alpha}(\nu,\xi) + \mathrm{otp}(A_\alpha(\tau)\setminus\xi) \mid \xi \in A_\alpha(\tau)\setminus\nu \,\}), \quad \text{if } c_A(\nu,\tau) \leq \alpha \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Lemma 2. If $\nu \leq \tau < \kappa^+$ then

$$d_{A,\alpha}(\nu,\tau) = \sup(\{\, d_{A,\alpha}(\nu,\xi) + \mathrm{otp}(A_\alpha(\tau)\setminus\xi) \mid \xi \in A_\alpha(\tau)\setminus\nu \text{ and } c_A(\nu,\xi) \leq \alpha \,\}),$$

i.e. $d_{A,\alpha}(\nu,\tau) = 0$ if and only if $\nu = \tau$ or $\alpha < c_A(\nu,\tau)$: ν and τ are not α-connected in A.

However, as will become apparent, this definition is in fact equivalent to measuring along a canonical long witness. Fix $A \subseteq \mathcal{F}$ and $\alpha < \kappa$ for the remainder of this section.

Lemma 3. If $\nu < \kappa^+$ and $\xi \in C_\alpha(\nu)$ then $d_{A,\alpha}(\xi,\nu) = \mathrm{otp}(A_\alpha(\nu)\setminus\xi)$.

Proof. The proof is by induction on $\zeta \in A_\alpha(\nu)\setminus\xi$. If $\zeta = \xi$ then $d_{A,\alpha}(\xi,\zeta) = 0 = \mathrm{otp}(A_\alpha(\zeta)\setminus\xi)$. Now suppose that ζ succeeds ζ' in $A_\alpha(\nu)\setminus\xi$. First of all note that by the hypothesis of the lemma and the case assumptions one has that $c_A(\xi,\nu)$, $c_A(\zeta,\nu)$ and $c_A(\zeta',\nu) \leq \alpha$, so one has that $c_A(\xi,\zeta) \leq \alpha$ by the subadditivity of c_A, Theorem (1.12). Also ζ, $\zeta' \in C_\alpha(\nu)$ as $\xi \in C_\alpha(\nu)$ and $C_\alpha(\nu)$ is a final segment of $A_\alpha(\nu)$ by Lemma (2.12). By Corollary (2.13) one has that $A_\alpha(\zeta)\setminus\xi = A_\alpha(\nu)\cap[\xi,\zeta)$ and $A_\alpha(\zeta')\setminus\xi = A_\alpha(\nu)\cap[\xi,\zeta')$. Hence $\mathrm{otp}(A_\alpha(\zeta)\setminus\tau) = \mathrm{otp}(A_\alpha(\zeta')\setminus\tau)+1$ for all $\tau \in (A_\alpha(\zeta)\setminus\xi)\cup\{\,\zeta'\,\} = A_\alpha(\zeta')\setminus\xi$. Thus

$$\begin{aligned} d_{A,\alpha}(\xi,\zeta) = \max\big(& d_{A,\alpha}(\xi,\zeta')+1, \\ & \sup(\{\, d_{A,\alpha}(\xi,\tau) + \mathrm{otp}(A_\alpha(\zeta')\setminus\xi) + 1 \mid \tau \in A_\alpha(\zeta')\setminus\xi \;\&\; c_A(\xi,\tau) \leq \alpha \,\})\big). \end{aligned}$$

But $\sup(\{\, d_{A,\alpha}(\xi,\tau) + \mathrm{otp}(A_\alpha(\zeta')\setminus\xi) + 1 \mid \tau \in A_\alpha(\zeta')\setminus\xi \;\&\; c_A(\xi,\tau) \leq \alpha \,\}) \leq$
$$\sup(\{\, d_{A,\alpha}(\xi,\tau) + \mathrm{otp}(A_\alpha(\zeta')\setminus\xi) \mid \tau \in A_\alpha(\zeta')\setminus\xi \;\&\; c_A(\xi,\tau) \leq \alpha \,\}) + 1 = d_{A,\alpha}(\xi,\zeta')+1,$$

so $d_{A,\alpha}(\xi,\zeta) = d_{A,\alpha}(\xi,\zeta') + 1$. By the induction hypothesis and Corollary (2.13) again this immediately gives that $d(\xi,\zeta) = \mathrm{otp}(A_\alpha(\zeta)\setminus\xi)$.

Finally, suppose that ζ is a limit point of $A_\alpha(\nu)\setminus\xi$. Then

$$\begin{aligned} d_{A,\alpha}(\xi,\zeta) &= \sup(\{\, d_{A,\alpha}(\xi,\tau) + \mathrm{otp}(A_\alpha(\zeta)\setminus\tau) \mid \tau\in A_\alpha(\zeta)\setminus\xi \;\;\&\;\; c_A(\xi,\tau)\le\alpha\,\}) \\ &= \sup(\{\,\mathrm{otp}(A_\alpha(\tau)\setminus\xi) + \mathrm{otp}(A_\alpha(\zeta)\setminus\tau) \mid \tau\in A_\alpha(\zeta)\setminus\xi\,\}) \end{aligned}$$

by the induction hypothesis, Lemma (2.12) and Corollary (2.13). Hence $d_{A,\alpha}(\xi,\zeta) = \mathrm{otp}(A_\alpha(\zeta)\setminus\xi)$ by the definition of $C_\alpha(\zeta)$ and Corollary (2.13).

Corollary 4. Lemma (3) says, in the notation of Definition (I.6), that if $\xi\in C_\alpha(\nu)$ then $d_{A,\alpha}(\xi,\nu) = \mathrm{otp}([\xi_\beta,\nu_\beta)) = D_\beta(\xi,\nu)$ if ψ^β_ν instantiates $A_\alpha(\nu)$.

Lemma 5. If $\nu<\kappa^+$, $\xi\le\rho$ and ξ, $\rho\in C'_\alpha(\nu)$ then

$$d_{A,\alpha}(\xi,\rho) = \mathrm{otp}(A_\alpha(\rho)\setminus\xi).$$

Proof. $\xi\in C_\alpha(\rho)$ by Lemma (2.19), so the conclusion is immediate from Lemma (3).

Lemma 6. If $\nu<\kappa^+$, $\xi\in C'_\alpha(\nu)$ and ψ^*_ν instantiates $A_\alpha(\eta)$ for all $\eta\in C'_\alpha(\nu)$ then

$$d_{A,\alpha}(\xi,\nu) = \mathrm{otp}\left(\mathrm{rge}(\psi^*_\nu)\cap\left[\xi,\mathrm{ssup}(A_\alpha(\nu))\right)\right).$$

Proof. If $C_\alpha(\nu)\ne\emptyset$ then $C'_\alpha(\nu) = C_\alpha(\nu)$, by Lemma (2.19). So ψ^*_ν instantiates $A_\alpha(\nu)$ and thus $\mathrm{rge}(\psi^*_\nu)\cap\left[\xi,\mathrm{ssup}(A_\alpha(\nu))\right) = A_\alpha(\nu)\setminus\xi$ and the conclusion is immediate from Lemma (3) again.

So suppose that $C_\alpha(\nu)=\emptyset$. As $\xi\in C'_\alpha(\nu)$ one has that $C'_\alpha(\nu)\setminus\xi = A_\alpha(\nu)\setminus\xi$ and that $c_A(\xi,\eta)\le\alpha$ if $\eta\in A_\alpha(\nu)\setminus\xi$, by Corollary (2.6) and the remark immediately after Definition (2.1), the definition of $A_\alpha(\eta)$. By Lemma (5) this gives that

$$d_{A,\alpha}(\xi,\nu) = \sup(\{\,\mathrm{otp}(A_\alpha(\eta)\setminus\xi) + \mathrm{otp}(A_\alpha(\nu)\setminus\eta) \mid \eta\in A_\alpha(\nu)\setminus\xi\,\}).$$

Recall that $A_\alpha(\nu)\cap[\xi,\eta)\subseteq A_\alpha(\eta)\setminus\xi$ if $\eta\in A_\alpha\setminus\xi$, by Corollary (2.6), so

$$\mathrm{otp}(A_\alpha(\eta')\setminus\xi) + \mathrm{otp}(A_\alpha(\nu)\setminus\eta') \le \mathrm{otp}(A_\alpha(\eta)\setminus\xi) + \mathrm{otp}(A_\alpha(\nu)\setminus\eta)$$

if η', $\eta\in A_\alpha(\nu)\setminus\xi$ and $\eta'<\eta$.

Thus if $A_\alpha\setminus\xi$ has a maximal element, say, ρ, then clearly

$$d_{A,\alpha}(\xi,\nu) = d_{A,\alpha}(\xi,\rho)+1 = \mathrm{otp}\left(\mathrm{rge}(\psi^*_\nu)\cap\left[\xi,\mathrm{ssup}(A_\alpha(\nu))\right)\right).$$

Now it is also clear from the definition of $d_{A,\alpha}$ that $d_{A,\alpha}(\xi,\eta)\le d_{A,\alpha}(\xi,\nu)$ for all $\eta\in A_\alpha(\nu)\setminus\xi$. Hence if $A_\alpha(\nu)\setminus\xi$ has no maximal element then

$$\begin{aligned} &\sup(\{\, d_{A,\alpha}(\xi,\eta) \mid \eta\in A_\alpha(\nu)\setminus\xi\,\}) = \sup(\{\,\mathrm{otp}(A_\alpha(\eta)\setminus\xi) \mid \eta\in A_\alpha(\nu)\setminus\xi\,\}) = \\ &\quad \mathrm{otp}\left(\mathrm{rge}(\psi^*_\nu)\cap\left[\xi,\mathrm{ssup}(A_\alpha(\nu))\right)\right) \le d_{A,\alpha}(\xi,\nu) \end{aligned}.$$

On the other hand, $\mathrm{otp}(A_\alpha(\eta) \setminus \xi) + \mathrm{otp}(A_\alpha(\nu) \setminus \eta) \leq \mathrm{otp}\left(\mathrm{rge}(\psi^*_\nu) \cap \left[\xi, \mathrm{ssup}(A_\alpha(\nu))\right)\right)$ for all $\eta \in A_\alpha(\nu) \setminus \xi$. Thus $d_{A,\alpha}(\xi, \nu) = \mathrm{otp}\left(\mathrm{rge}(g) \cap \left[\xi, \mathrm{ssup}(A_\alpha(\nu))\right)\right)$ as required.

Theorem 7. Suppose $c_A(\nu, \tau) \leq \alpha$ and let $\langle \zeta_i \mid i \leq l\rangle$ and $\langle (\psi^*_{\zeta_i}) \mid i < l\rangle$ be canonical long witnesses to this. (Recall: $\nu = \zeta_l$, $\tau = \zeta_0$.) Write γ_i for $\gamma^*(\zeta_i)$, ψ_i for $\psi^*_{\zeta_i}$ and ζ^*_i for $\mathrm{ssup}(A_\alpha(\zeta_i))$. Then

$$\begin{aligned} d_{A,\alpha}(\nu, \tau) = &\ \mathrm{otp}\left(\mathrm{rge}(\psi_{l-1}) \cap [\nu, \zeta^*_{l-1})\right) + \mathrm{otp}\left(\mathrm{rge}(\psi_{l-2}) \cap [\zeta_{l-1}, \zeta^*_{l-2})\right) + \\ &\ \ldots + \mathrm{otp}\left(\mathrm{rge}(\psi_1) \cap [\zeta_2, \zeta^*_1)\right) + \mathrm{otp}\left(\mathrm{rge}(\psi_0) \cap [\zeta_1, \tau^*)\right). \\ = &\ D_{\gamma_{l-1}}(\nu, \zeta^*_{l-1}) + \ldots + D_{\gamma_1}(\zeta_2, \zeta^*_1) + D_{\gamma_0}(\zeta_1, \tau^*). \end{aligned}$$

Proof. The proof is by induction on l. (Recall that l is determined by ν and τ if $c_A(\nu, \tau) \leq \alpha$, so this is a well defined proof strategy!) For $l = 1$ the theorem is immediate from Lemmas (3) and (6). If $l = l' + 1$ then by the theorem for l' applied to ν, ζ_1 and Lemma (2.31) one has that $d_{A,\alpha}(\nu, \xi) + \mathrm{otp}(A_\alpha(\tau) \cap [\xi, \zeta_1)) \leq d_{A,\alpha}(\nu, \zeta_1)$ for any $\xi \in A_\alpha(\tau) \cap [\nu, \zeta_1]$, so in order to calculate $d(\nu, \tau)$ one need only calculate the supremum of the $d_{A,\alpha}(\nu, \xi) + \mathrm{otp}(A_\alpha(\tau) \cap [\xi, \zeta_1))$ for $\xi \in A_\alpha(\tau) \cap [\zeta_1, \tau)$. If $C_\alpha(\tau) \neq \emptyset$ the theorem now follows from calculating that $d_{A,\alpha}(\nu, \xi) = d_{A,\alpha}(\nu, \zeta_1) + \mathrm{otp}(A_\alpha(\tau) \setminus \zeta_1)$ by induction along $C_\alpha(\tau) \neq \emptyset$ exactly as in the proof of Lemma (3), while if $C_\alpha(\tau) = \emptyset$ the theorem follows as in the proof of Lemma (6).

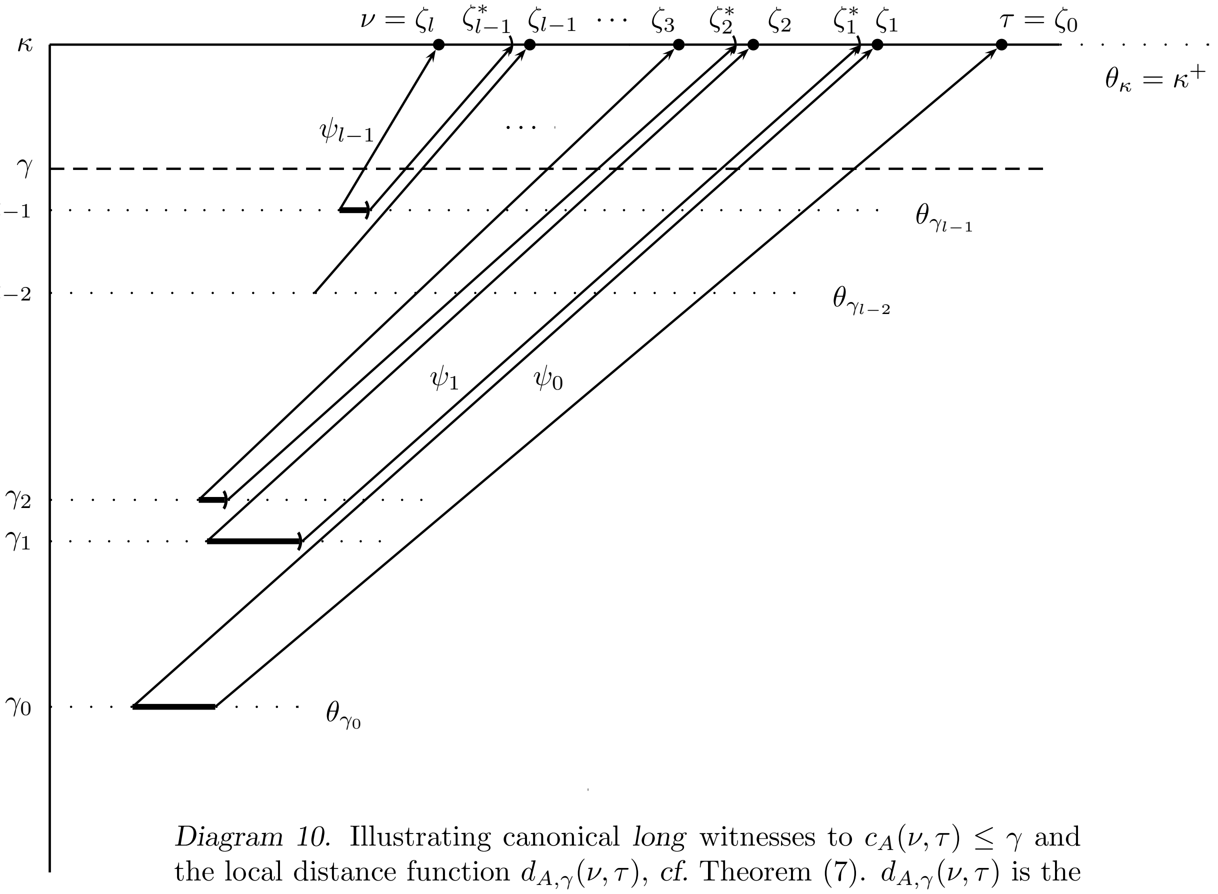

Diagram 10. Illustrating canonical *long* witnesses to $c_A(\nu, \tau) \leq \gamma$ and the local distance function $d_{A,\gamma}(\nu, \tau)$, *cf.* Theorem (7). $d_{A,\gamma}(\nu, \tau)$ is the sum, reading from top to bottom. of the ordertypes of the boldface lines,

Corollary 8. ([K00, Fact 2.13(3)] for $\kappa = \omega_1$ and A finite.) If $\nu < \eta < \tau < \kappa^+$ and $c_A(\nu, \eta)$, $c_A(\eta, \tau) \leq \alpha$ then

$$d_{A,\alpha}(\nu, \tau) = d_{A,\alpha}(\nu, \eta) + d_{A,\alpha}(\eta, \tau).$$

Proof. Immediate from Theorem (7) and Lemma (2.30).

I draw a couple more simple corollaries of Theorem (7).

Corollary 9. Suppose $\nu < \tau < \kappa^+$ and $\alpha < \kappa$. Let $A \subseteq B \subseteq \mathcal{F}$ and suppose whenever $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ are sequences witnessing that $c_B(\nu, \tau) \leq \alpha$ there are $\langle (\beta_i, f_i) \mid i < k \rangle$ with $\gamma_i \leq \beta_i$ for all $i < k$ and such that $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\beta_i, f_i) \mid i < k \rangle$ witness that $c_A(\nu, \tau) \leq \alpha$. (In words: whenever one has witnesses that $c_B(\nu, \tau) \leq \alpha$ one can replace any maps used from $B \setminus A$ by maps in A which are start at least as far up in the simplified morass.) Then $d_{A,\alpha}(\nu, \tau) = d_{B,\alpha}(\nu, \tau)$.

Proof. Immediate from Theorem (7) and Definition (2.16), the definition of the $C'_\alpha(\nu)$.

Corollary 10. Suppose $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \tau < \kappa^+$. Suppose $c_A(\nu, \tau) = \beta < \alpha$ and $\{\gamma \mid \exists (\gamma, g) \in A\ \exists$ a witnessing sequence to $c_A(\nu, \tau) < \beta$ in which (γ, g) is used$\} \cap [\beta, \alpha) = \emptyset$. Then $d_{A,\gamma}(\nu, \tau) = d_{A,\beta}(\nu, \tau)$ for all $\gamma \in [\beta, \alpha)$.

Proof. Immediate from Theorem (7) and Definition (2.16), the definition of the $C'_\alpha(\nu)$.

Remark 11. It is trivial from Theorem (7) that for all $A \subseteq B \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \tau < \kappa^+$ one has that $d_{A,\alpha}(\nu, \tau) \leq d_{B,\alpha}(\nu, \tau)$.

While $d_{A,\alpha}(\nu, \tau) \leq D_\alpha(\nu, \tau)$ there are reasons for considering another distance function.

Definition 12. Suppose $A \subseteq \mathcal{F}$ and $\alpha < \kappa$. For $\nu < \tau < \kappa^+$ set

$$\mathfrak{d}_{A,\alpha}(\nu, \tau) = \sup(\{d_{B,\alpha}(\nu, \tau) \mid B \in [A]^{<\omega}\}).$$

Note 13. By Remark (11) it is clear that for $A \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \tau < \kappa^+$ one has $\mathfrak{d}_{A,\alpha}(\nu, \tau) \leq d_{A,\alpha}(\nu, \tau)$.

It is easy to see that $\mathfrak{d}_{A,\alpha}$ shares many useful properties with $d_{A,\alpha}$, in particular the conclusions of Corollaries (8)-(10) and Remark (11). (In order to prove these properties of $\mathfrak{d}$ one only needs Theorem (7) and Corollary (8) for finite A.)

Lemma 14. $A \subseteq B \subseteq \mathcal{F}$, $\alpha < \kappa$ and $\nu < \tau < \kappa^+$ one has that $\mathfrak{d}_{A,\alpha}(\nu, \tau) \leq \mathfrak{d}_{B,\alpha}(\nu, \tau)$.

Proof. Immediate from Remark (11).

Lemma 15. If $A \subseteq \mathcal{F}$, $\alpha < \kappa$, $\nu < \eta < \tau < \kappa^+$ and $c_A(\nu, \eta)$, $c_A(\eta, \tau) \leq \alpha$ then

$$\mathfrak{d}_{A,\alpha}(\nu, \tau) = \mathfrak{d}_{A,\alpha}(\nu, \eta) + \mathfrak{d}_{A,\alpha}(\eta, \tau).$$

Proof. For each $B \in [A]^{<\omega}$ one has that $d_{B,\alpha}(\nu, \tau) = d_{B,\alpha}(\nu, \eta) + d_{B,\alpha}(\eta, \tau)$ by Corollary (8). Thus $\mathfrak{d}_{A,\alpha}(\nu, \tau) \leq \mathfrak{d}_{A,\alpha}(\nu, \eta) + \mathfrak{d}_{A,\alpha}(\eta, \tau)$.

On the other hand, $\mathfrak{d}_{A,\alpha}(\nu,\eta)+\mathfrak{d}_{A,\alpha}(\eta,\tau)=\sup(\{d_{C,\alpha}(\nu,\eta)+d_{D,\alpha}(\eta,\tau)\mid C,\,D\in[A]^{<\omega}\})$. If $C,\,D\in[A]^{<\omega}$ and $B=C\cup D$ then $B\in[A]^{\omega}$ and one has that $d_{C,\alpha}(\nu,\eta)\le d_{B,\alpha}(\nu,\eta)$ and $d_{D,\alpha}(\eta,\tau)\le d_{B,\alpha}(\eta,\tau)$. Thus $d_{C,\alpha}(\nu,\eta)+d_{D,\alpha}(\eta,\tau)\le d_{B,\alpha}(\nu,\eta)+d_{B,\alpha}(\eta,\tau),\;=d_{B,\alpha}(\nu,\tau)$, by Corollary (8). Hence $\mathfrak{d}_{A,\alpha}(\nu,\eta)+\mathfrak{d}_{A,\alpha}(\eta,\tau)\le\mathfrak{d}_{A,\alpha}(\nu,\tau)$.

Lemma 16. Suppose $\nu<\tau<\kappa^{+}$ and $\alpha<\kappa$. Let $A\subseteq B\subseteq\mathcal{F}$ and suppose whenever $\langle\xi_i\mid i\le k\rangle$ and $\langle(\gamma_i,g_i)\mid i<k\rangle$ are sequences witnessing that $c_B(\nu,\tau)\le\alpha$ there are $\langle(\beta_i,f_i)\mid i<k\rangle$ with $\gamma_i\le\beta_i$ for all $i<k$ and such that $\langle\xi_i\mid i\le k\rangle$ and $\langle(\beta_i,f_i)\mid i<k\rangle$ witness that $c_A(\nu,\tau)\le\alpha$. (In words: whenever one has witnesses that $c_B(\nu,\tau)\le\alpha$ one can replace any maps used from $B\setminus A$ by maps in A which are start at least as far up in the simplified morass.) Then $\mathfrak{d}_{A,\alpha}(\nu,\tau)=\mathfrak{d}_{B,\alpha}(\nu,\tau)$.

Proof. By the hypothesis, if $b\in[B]^{<\omega}$ there is some $a\in[A]^{<\omega}$ such that $d_{b,\alpha}(\nu,\tau)\le d_{a,\alpha}(\nu,\tau)$. Hence $\mathfrak{d}_{B,\alpha}(\nu,\tau)\le\mathfrak{d}_{A,\alpha}(\nu,\tau)$. Thus, by Lemma (14), $\mathfrak{d}_{A,\alpha}(\nu,\tau)=\mathfrak{d}_{B,\alpha}(\nu,\tau)$.

Lemma 17. Suppose $A\subseteq\mathcal{F}$, $\alpha<\kappa$ and $\nu<\tau<\kappa^{+}$. Suppose $c_A(\nu,\tau)=\beta<\alpha$ and $\{\gamma\mid\exists(\gamma,g)\in A\;\exists$ a sequence witnessing $c_A(\nu,\tau)<\beta$ in which (γ,g) is used$\}\cap[\beta,\alpha)=\emptyset$. Then $\mathfrak{d}_{A,\gamma}(\nu,\tau)=\mathfrak{d}_{A,\beta}(\nu,\tau)$ for all $\gamma\in[\beta,\alpha)$.

Proof. Set $B=\{(\varepsilon,g)\in A\mid\varepsilon\le\beta\}$. The conclusion is now immediate from Lemma (16) since $\mathfrak{d}_{A,\beta}=\mathfrak{d}_{B,\gamma}$ and $B\subseteq A$.

One advantage of $\mathfrak{d}_{A,\alpha}$ over $d_{A,\alpha}$ is that it is better adapted for closure arguments.

Lemma 18. Let $\alpha<\kappa$ and $\nu<\tau<\kappa^{+}$. Suppose that $\langle A_i\mid i<\xi\rangle$ is a chain of subsets of $\mathcal{F}$, so that $A_i\subseteq A_j\subseteq\mathcal{F}$ for $i<j<\xi$. Let $A=\bigcup\{A_i\mid i<\xi\}$. Then $\mathfrak{d}_{A,\alpha}(\nu,\tau)=\mathrm{ssup}(\{\mathfrak{d}_{A_i,\alpha}\mid i<\xi\})$.

Proof. If the sequence is eventually constant the conclusion is trivial. If not then ξ is a limit ordinal and whenever $B\in[A]^{<\omega}$ there is some $i<\xi$ such that $B\in[A_i]^{<\omega}$. Hence $d_{B,\alpha}(\nu,\tau)\le\mathfrak{d}_{A_i,\alpha}(\nu,\tau)$, and so $\mathfrak{d}_{A,\alpha}(\nu,\tau)\le\sup(\{\mathfrak{d}_{A_i,\alpha}\mid i<\xi\})$. Thus $\mathfrak{d}_{A,\alpha}(\nu,\tau)=\mathrm{ssup}(\{\mathfrak{d}_{A_i,\alpha}\mid i<\xi\})$, by Lemma (14).

§4. Using c_A in forcing arguments.

Throughout this section let κ be an arbitrary successor cardinal with cardinal predecessor μ, so that $\mu=\kappa^{-}$ and $\kappa=\mu^{+}$. Also suppose that throughout this section that $\mathbb{M}$ is stationary (*cf.* Definition (1.11)).

In the section I examine the functions c_A in the context of abstract arguments about κ-$\mathbb{M}$-proper forcing when κ is an arbitrary successor cardinal, concentrating particularly on proving general properties of the functions c_A which help in proofs of κ-$\mathbb{M}$-properness. The results of §1 are used extensively, but not those of §§2,3. I start, however, with a general and well-known general fact about cardinal preservation.

Definition 1. Let $\mathbb{P}$ be forcing notion. Suppose that $\mathbb{P}\in H_\lambda$ for some regular cardinal λ. Let $(\mathcal{N},\epsilon)\prec(H_\lambda,\in)$ with $\overline{\overline{\mathcal{N}}}=\mu$, $\mu\subseteq\mathcal{N}\cap\kappa\in\kappa$ and $\mathbb{P}\in\mathcal{N}$. Then $p^*\in\mathbb{P}$ is $(\mathbb{P},\mathcal{N})$-*generic* if whenever $\mathcal{D}\in\mathcal{N}$ is a dense and open subset of $\mathbb{P}$, $q\in\mathcal{D}$ and $q\le p^*$ then there is some $s\in\mathcal{D}\cap\mathcal{N}$ compatible with q, *i.e.* such that there is some $r\in\mathbb{P}$ such that $r\le q,\,s$.

Fact 2. (Well known folklore.) If for every $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{f}$ with $p \Vdash \dot{f} : \mu \longrightarrow \kappa$ there is some $(\mathcal{N}, \epsilon) \prec (H_\lambda, \in)$ with $\overline{\overline{\mathcal{N}}} = \mu$, $\mu \subseteq \mathcal{N} \cap \kappa \in \kappa$, $\dot{f}$, p, $\mathbb{P} \in \mathcal{N}$, and some $p^* \leq p$ which is $(\mathbb{P}, \mathcal{N})$-generic then $\Vdash_{\mathbb{P}} \mu^+ = \kappa$.

Proof. This is just as the usual proof that properness preserves ω_1. Let $\dot{f}$, p, $\mathbb{P}$ be such that $p \Vdash \dot{f} : \mu \longrightarrow \kappa$. Let $\mathcal{N}$ be as in the statement of the fact. Let p^* be $(\mathbb{P}, \mathcal{N})$-generic. For each $\alpha < \mu$ let $\mathcal{D}_\alpha = \{ p' \leq p \mid \exists \gamma_{p'} \in \mathrm{On}\ p' \Vdash \text{“}\dot{f}(\alpha) = \hat{\gamma}_{p'}\text{”} \}$. Clearly each $\mathcal{D}_\alpha$ is a dense and open subset of $\mathbb{P}$. Let A_α be a maximal antichain in $\mathbb{P}$ for each $\alpha < \mu$. Suppose that $x \in A_\alpha$, so $x \leq p$, and suppose that x is compatible with p^*. Let $D'_\alpha = \{ y \in D_\alpha \mid \exists z \in A_\alpha\ y \leq z \}$. Clearly D'_α is dense and open. Let $q \in \mathcal{D}'_\alpha$ be such that $q \leq x$, p^*. Then there is some $s \in \mathcal{D}'_\alpha \cap \mathcal{N}$ and some $t \in \mathbb{P}(\cap \mathcal{D}'_\alpha)$ with $t \leq s$, q. As $t \leq x$ one has that $s \leq x$ (otherwise t would be below two (incompatible) elements of A_α). But $s \Vdash \text{“}\dot{f}(\alpha) \in \mathcal{N} \cap \kappa\text{”}$ since $s \in \mathcal{N}$. So $x \Vdash \text{“}\dot{f}(\alpha) \in \mathcal{N} \cap \kappa\text{”}$ as well. Thus one also has that $p^* \Vdash \text{“}\dot{f}(\alpha) \in \mathcal{N} \cap \kappa\text{”}$.

This shows that $\{ w \in \mathbb{P} \mid \exists \delta < \kappa\ w \Vdash \dot{f}(\alpha) < \delta \}$ is dense below any p such that $p \Vdash \dot{f} : \mu \longrightarrow \kappa$. Hence $\Vdash_{\mathbb{P}} \mu^+ = \kappa$.

Fact (2) has been known for a considerable time — for example it is implicit in the brief remark after [AShe, Theorem 3] that (what is essentially) Baumgartner's forcing for adding a club subset of ω_1 with finite conditions “can be easily generalised to the case $\kappa = \mu^+ \ldots$ without adding sets of size $< \mu$ and without collapsing cardinals.” Of course most attention has been paid to the case $\kappa = \omega_1$, because one can prove iteration theorems for proper forcing, while this is much harder in the general case (*cf.* [S*]). However, as one is primarily interested in using κ-$\mathbb{M}$-properness to help guarantee cardinal preservation for single-step forcings this problem is not important.

Definition 3. Let $\mathbb{P}$ be forcing notion with $\mathbb{P} \in H_\lambda$ for some regular cardinal λ. Call $\mathcal{N}$ a *good (sub) model (of H_λ)* if $(\mathcal{N}, \epsilon) \prec (H_\lambda, \in)$, $\overline{\overline{\mathcal{N}}} = \mu$, $\{ \mathbb{M}, \mathcal{F}, c, \kappa, \kappa^+ \} \cup \mu \subseteq \mathcal{N}$, $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \kappa \in \kappa$, there is some $F \in \mathcal{F}_{\delta\kappa}$ such that $\mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$, and for each $(\alpha, f) \in \mathcal{F}$ with $\alpha < \delta$ if there is some $f' \in \mathcal{F}_{\alpha\delta}$ such that $f = F \cdot f'$ then $(\alpha, f) \in \mathcal{N}$. $\mathbb{P}$ is κ-$\mathbb{M}$-*proper* if there is some $x \in [H_\lambda]^\mu$, such that whenever $p \in \mathbb{P}$ and $\mathcal{N}$ is a good model with $\{ p, \mathbb{P} \} \cup x \subseteq \mathcal{N}$ there is some $p^* \leq p$ which is $(\mathbb{P}, \mathcal{N})$-generic.

Note that if $\mathcal{N}$ is good and $(\alpha, f) \in \mathcal{N}$ then one has that there is some $f' \in \mathcal{F}_{\alpha\delta}$ such that $f = F \cdot f'$.

Lemma 4. If $\mathbb{P}$ is κ-$\mathbb{M}$-proper then $\Vdash_{\mathbb{P}} \mu^+ = \kappa$.

Proof. Given any $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{f}$ such that $p \Vdash \dot{f} : \mu \longrightarrow \kappa$ let $\mathcal{N}$ be as in the definition of $\mathbb{M}$-κ-proper with $\dot{f} \in \mathcal{N}$. There are a closed unbounded in $[H_\lambda]^\mu$ set of such $\mathcal{N}$ without the requirement on F, and stationarily many with it by $\mathbb{M}$'s stationarity. The conclusion of the lemma follows from Fact (2) immediately.

I concentrate on a particular type of forcings analogous to forcings with side conditions which are matrices of models. Conditions in these forcings have *working parts* with an associated *realm*, a subset of size less than μ of κ^+, which is the part of the domain (an unbounded subset of κ^+) which it will contribute to any generic object to which it belongs. They also have *side condition* parts which are some $A \in [\mathcal{F}]^{<\mu}$. So I start by considering pairs $p = (a_p, A_p)$ with $a_p \in [\kappa^+]^{<\mu}$ and

$A_p \in [\mathcal{F}]^{<\mu}$. In analysing these forcings below it is convenient to fix some pieces of notation and working assumptions as one goes. These pieces of notation and assumptions are flagged with a $(\star)$.

A number of the basic results below are extracted from Koszmider's proof in [K00] of $\mathbb{M}$-properness for a forcing to add a chain of length ω_2 in ${}^{\omega_1}\omega_1/\mathit{fin}$. More accurately, these results are abstracted from a construction to add a chain of length ω_2 in $\mathcal{P}(\omega_1)/\mathit{fin}$ using $\mathbb{M}$-proper forcing. Although in [K98a] this is done via a ccc forcing using a stepping up function, Koszmider revealed in personal communication that his original proof used (ω_1-)$\mathbb{M}$-proper forcing, and, indeed, such a construction can reasonably easily be read off from [K00] with the aid of [K98a]. The main technical difference between this construction and that in [K00] is that there is no need for the local distance functions $d_{A,\alpha}$ discussed in §3 in order to add a chain of length ω_2 in $\mathcal{P}(\omega_1)/\mathit{fin}$. Conditions in Koszmider's forcings, for example, have a working part and a side condition part components as discussed above. The working parts have form $F_p : a_p \times b_p \longrightarrow 2$ or ω_1, where a_p is the realm of the condition and $b_p \in [\omega_1]^{<\omega}$.

Definition 5. Let $\mathbb{P}_0$ be $\{ p \,|\, p = (a_p, A_p)$ where $a_p \in [\kappa^+]^{<\mu}$ and $A_p \in [\mathcal{F}]^{<\mu} \}$ ordered by reverse inclusion.

Of course, any two conditions in $\mathbb{P}_0$ are compatible since if p, $q \in \mathbb{P}_0$ then $(a_p \cup a_q, A_p \cup A_q) \leq p$, q, so forcing with $\mathbb{P}_0$ itself is uninteresting and preserves all cardinals. However this is not the case for more elaborate forcing notions whose conditions merely have retracts in $\mathbb{P}_0$.

$(\star)$ Suppose that $\mathbb{P}$ is a forcing and that conditions in $\mathbb{P}$ are of the form $p = (a_p, A_p, \ldots)$ where $(a_p, A_p) \in \mathbb{P}_0$, a_p is the realm of p and A_p is the side condition part of p.

$(\star)$ Fix $p \in \mathbb{P}$. Suppose that $\mathcal{N} \prec H_{\kappa^{++}}$ is good with $(\delta, F) \in \mathcal{F}$ such that $\mathcal{N} \cap \kappa^+ = \mathrm{rge}(F)$.

In order to prove that $\mathbb{P}$ is κ-$\mathbb{M}$-proper the typical strategy, and the only one which will be dealt with in this paper, is to show that $(a_p, A_p \cup \{ (\delta, F) \}, \ldots)$ is $(P, \mathcal{N})$-generic. In practice one also has to show that this is a condition and is stronger than p, but here the aim is to analyse the situation in which this is possible, so simply assume that it is so.

$(\star)$ Assume that $(a_p, A_p \cup \{ (\delta, F) \}, \ldots) \in \mathbb{P}$ and that $(a_p, A_p \cup \{ (\delta, F) \}, \ldots) \leq p$.

Remark 6. One may as well assume that $\mathcal{N}$ satisfies the following

(i) If $(\alpha, f) \in \mathcal{N}$ then $\exists f' \in \mathcal{F}_{\alpha\delta}\ (f = F \cdot f')$.

(ii) If $\alpha < \delta$, $g \in \mathcal{F}_{\alpha\kappa}$, $g' \in \mathcal{F}_{\alpha\delta}$, $g'' \in \mathcal{F}_{\delta\kappa}$ and $g = g'' \cdot g'$ then $h(g) = F \cdot g' \in \mathcal{F}_{\alpha\kappa} \cap \mathcal{N}$ (and $(\alpha, h(g)) \in \mathcal{F} \cap \mathcal{N}$).

Proof. These remarks are standard and follow from the stationarity of $\mathbb{M}$. By the stationarity of $\mathbb{M}$ one may as well assume that one has an $\mathcal{N}$ for which properties such as these hold since, eg, $\mathcal{N} \cap \kappa^+$ is club in $\mathcal{P}_{\leq\mu}(\kappa^+)$. The point is that if $j : \overline{\mathcal{N}} \longrightarrow \mathcal{N}$ is the inverse of the transitive collapse of $\mathcal{N}$ then $j^{-1} \restriction \mathcal{N} \cap \kappa^+ = F^{-1} \restriction \mathcal{N} \cap \kappa^+$ and if $j(\langle \langle \overline{\theta}_\alpha \,|\, \alpha \leq \delta \rangle, \langle \overline{\mathcal{F}}_{\alpha\beta} \,|\, \alpha \leq \beta \leq \delta \rangle \rangle) = \mathbb{M}$ then $\langle \langle \overline{\theta}_\alpha \,|\, \alpha \leq \delta \rangle, \langle \overline{\mathcal{F}}_{\alpha\beta} \,|\, \alpha \leq \beta \leq \delta \rangle \rangle = \langle \langle \theta_\alpha \,|\, \alpha \leq \delta \rangle, \langle \mathcal{F}_{\alpha\beta} \,|\, \alpha \leq \beta \leq \delta \rangle \rangle$.

$(\star)$ Suppose that $\mathcal{D}$ is a dense, open subset of $\mathbb{P}$ with $\mathcal{D} \in \mathcal{N}$. Suppose also that $q \in \mathcal{D}$ and that $q \leq (a_p, A_p \cup \{ (\delta, F) \}, \ldots)$. Let $q \restriction \mathcal{N} = q \cap \mathcal{N} = (a_q \cap \mathcal{N}, A_q \cap \mathcal{N}, \ldots)$.

In practice one now needs to show that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and that $q \leq q \restriction \mathcal{N}$, and this may be a challenging part of the proof of κ-$\mathbb{M}$-properness. Again, here the aim is to analyse the situation in which this is possible, so simply assume that this can be done.

($\star$) $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and $q \leq q \restriction \mathcal{N}$.

If one wants to show that $(a_p, A_p \cup \{ (\delta, F) \}, \ldots)$ is $(\mathbb{P}, \mathcal{N})$-generic it is typically helpful to explicitly reflect some of the properties of q and of the relationship between q and $q \restriction \mathcal{N}$ to $\mathcal{N}$ so that one can get an $s \in \mathcal{N} \cap \mathcal{D}$ which one can actually prove is amalgamable with q. In order to chrystalise these ideas a slew of notation and definitions, given in Notation (7) below, is unavoidable.

It would also be useful if one could without loss of generality smooth out A_q, by adding some further maps if necessary, in order to make its internal structure more uniform. This would be useful because the uniformity together with its reflection to s might also be helpful in performing the amalgamation of q and s. One can do this smoothing out for the partial order $\mathbb{P}_0$, since $\mathcal{D}$ is dense and open, and if $t \in \mathbb{P}_0$ and $A_t \subseteq A \subseteq \mathcal{F}$ then $(a_t, A) \leq t$, and so adding maps to A_q will give a stronger condition and also not take one outside $\mathcal{D}$. But $\mathbb{P}_0$ is deceptive because one of the essential points of κ-$\mathbb{M}$-proper forcing, as of proper forcing with chains of models as side conditions, is that typically one will give constraints on the working part of a condition which are dependent on the interaction between its realm and the side condition part of the condition. Thus one cannot simply throw extra maps into the side condition part of a condition willy-nilly and expect that conflicts with these constraints will not arise. It is likely that one will end up either with a condition that is not stronger than the one with which one started or with something that is not a condition at all.

However, fortunately, in the cases that have been looked at so far it has been possible to throw *some* additional maps into A_q and get a condition stronger than q.

The minimal amount of smoothing that seems to be necessary to get a satisfactory theory is to ensure that one has $c_{A_q}(\nu, \tau) < \kappa$ for all ν, $\tau \in a_q$. Stronger conditions that may be useful are that there is some $(\alpha, f) \in A_p$ such that $a_p \subseteq \operatorname{rge}(f)$, or that

$$\exists (\alpha, f) \in A_p \ \big(\big(\forall (\beta, g) \in A_p \setminus \{ (\alpha, f) \} \ (\beta < \alpha \ \ \& \ \ \exists h \in \mathcal{F}_{\beta\alpha} \ g = f \cdot h)\big) \ \ \& \ \ a_p \subseteq \operatorname{rge}(f) \big).$$

These conditions *have* been achievable in the cases considered so far. (The point is that one can factor all of the maps in A_q through some single map $(\alpha, f) \in \mathcal{F}$ with α arbitary large below κ by Fact (I.4.b) and certainly above $c_{A_q}(\nu, \tau) < \alpha$ for all ν, $\tau \in a_q$ with $c_{A_q}(\nu, \tau) < \kappa$, and add this map to A_q. This reduces $c_{A_q}(\nu, \tau)$ (to α) for ν, $\tau \in a_q$ if (and only if) it previously took the value κ and this tends to be benign if α is chosen large enough.)

($\star$) Consequently I assume from now onwards that $c_{A_q}(\nu, \tau) < \kappa$ for all ν, $\tau \in a_q$.

The following are three examples of the sort of uniformities in A_q that might conceivably be useful but for which it is harder to check that they can be obtained. I do not know of any occasions in which they have been checked and used (and the referee of this paper commented that to them they "seem bizarre"!).

(i) $\quad \forall (\alpha, f), (\alpha, f'), (\beta, g) \in A_q \big((\beta < \alpha \ \ \& \ \ \exists h \in \mathcal{F}_{\beta\alpha} \ g = f \cdot h) \longrightarrow (\beta, f' \cdot h \in A_q) \big),$

(ii) $\quad \forall (\alpha, f), (\beta, g) \in A_q \ \big(\beta < \alpha \longrightarrow \exists f' \in \mathcal{F}_{\alpha\kappa} \ \exists h \in \mathcal{F}_{\beta\alpha} \ g = f' \cdot h \ \ \& \ \ (\alpha, f') \in A_q \big),$

(iii) $\quad \exists(\alpha,f)\in(A_q\cap\mathcal{N})\big(\alpha<\delta\ \ \&\ \ \exists h\in\mathcal{F}_{\alpha\delta}\ (f=F\cdot h\ \ \&\ \ a_q\cap\mathcal{N}\subseteq \mathrm{rge}(f)\ \ \&$
$\qquad \forall(\beta,g)\in(A_q\cap\mathcal{N})\setminus\{(\alpha,f)\}\ (\beta<\alpha\ \ \&\ \ \exists h\in\mathcal{F}_{\beta\alpha}\ g=f\cdot h))\big).$

I now introduce the promised notation. (The definition of Y_i follows Koszmider in [K00] for $\kappa=\omega_1$ and a_q, A_q finite.) Recall, from Remark (6) that if $(\alpha,f)\in\mathcal{F}$, $\alpha<\delta$ and $f=f''\cdot f'$ where $(\delta,f'')\in\mathcal{F}$ and $f'\in\mathcal{F}_{\alpha\delta}$ then $h(f)=F\cdot f'$.

Notation 7. Let $\langle(\beta_i,f_i)\,|\,i<\chi\rangle$ enumerate $\langle(\alpha,f)\in A_q\,|\,\alpha<\delta\rangle$ for some $\chi<\mu$.

Let $\quad Y_i=\mathrm{rge}(f_i)\cap\mathrm{rge}(h(f_i))$ for $i<\chi$.

Note that $\quad Y_i=\mathrm{rge}(f_i)\cap\mathrm{rge}(F)$ for all $i<\chi$ as well.

Let $\quad \rho_i=\mathrm{ssup}(\{\rho<\theta_{\alpha_i}\,|\,f_i(\rho)=h(f_i)(\rho)\})$.

Then $\quad Y_i=\mathrm{rge}(f_i\restriction\rho_i)=\mathrm{rge}(\psi_{(\alpha_i,\rho_i),(\kappa,f_i(\rho_i))}\restriction\rho_i)$.

Let $\quad \beta^*=\mathrm{ssup}(\{\beta_i\,|\,i<\chi\})$.

Let $\quad b_q=\{c_{A_q}(\zeta,\xi)\,|\,\zeta,\xi\in a_q\}=c_{A_q}``[a_q]^2$.

Let $\quad e_q\in[\kappa]^{<\mu}$ be arbitrary, $e_{q\restriction\mathcal{N}}=e_q\cap\mathcal{N}$, and $\beta^\dagger=\max(\{\beta^*,\mathrm{ssup}(e_{q\restriction\mathcal{N}})\})$.

As $\overline{\overline{Y_i}}$ is an initial segment of $\mathrm{rge}(h(f_i))\subseteq\mathrm{rge}(F)=\mathcal{N}$ for $i<\chi$ (by Fact (I.3)), one has that $Y_i\subseteq\mathcal{N}$ for each $i<\chi$. Note also that $Y_i\in\mathcal{N}$ and as $\mathcal{N}^{<\mu}\subseteq\mathcal{N}$ one has that $\langle Y_i\,|\,i<\chi\rangle\in\mathcal{N}$.

(Koszmider ([K00]), incidentally, also found the following notational definitions useful in connection with properties of $d_{A_q,\alpha}$ for various α. They will not be used in the remainder of the paper, so the reader may wish to skip over them.

Let $\quad \{\xi_i\,|\,i<\lambda\}$ enumerate $(a_q\setminus\mathcal{N})\cap\sup(\mathcal{N}\cap\kappa^+)$.

Let $\quad \eta_i=\min(\mathcal{N}\setminus\xi_i)$ and $\nu_i=\sup(\{\sup(Y_j\cap\eta_i)\,|\,j<\chi\})$ for $i<\lambda$.)

Let $\phi(x)$ be the conjunction of the following.

(i) $\quad x\in\mathcal{D}$, (ii) $x\le q\restriction\mathcal{N}$, and (iii) $\exists(\alpha^*,h^*)\in A_x\ \ \beta^\dagger<\alpha^*$ such that

(1) $\quad \forall(\beta,f)\in A_x\ \big(\beta<\beta^*\longrightarrow\exists i<\chi\ (\mathrm{rge}(f)\cap\mathrm{rge}(h^*)=Y_i)\big)$,

(2) $\quad \forall i<\chi\ \exists(\beta_i,f)\in A_x\ \big(\mathrm{rge}(f)\cap\mathrm{rge}(h^*)=Y_i\ \ \&\ \ \exists f''\in\mathcal{F}_{\alpha^*\kappa}\ \exists f'\in\mathcal{F}_{\beta\alpha^*}$
$\qquad f=f''\cdot f'\longrightarrow\mathrm{rge}(h^*\cdot f')\cap\mathrm{rge}(f)=Y_i\big)$,

(3) $\quad a_{q\restriction\mathcal{N}}\subseteq\mathrm{rge}(h^*)$,

(4) $\quad (a_x\setminus a_{q\restriction\mathcal{N}})\cap\mathrm{rge}(h^*)=\emptyset$,

(5) $\quad e_{q\restriction\mathcal{N}}\subseteq e_x\ \ \&\ \ (e_x\setminus\beta^\dagger)\cap\alpha^*=\emptyset$,

(6) $\quad \forall\gamma\in b_x\ \gamma<\kappa$, and

The following sentences would correspond to the stronger conditions on A_q mentioned above, but are *not* part of the definition of ϕ.

(α) $\quad \exists(\alpha,f)\in A_x\ \big(\forall(\beta,g)\in A_x\setminus\{(\alpha,f)\}\ (\beta<\alpha\ \ \&\ \ \exists k\in\mathcal{F}_{\beta\alpha}\ g=f\cdot k)\ \ \&\ \ a_x\subseteq\mathrm{rge}(f)\big)$

(β) $\quad \forall(\alpha, f),\, (\alpha, f'),\, (\beta, g) \in A_x\, \big((\beta < \alpha \;\;\&\;\; \exists k \in \mathcal{F}_{\beta\alpha}\; g = f \cdot k) \longrightarrow (\beta, f' \cdot k) \in A_x\big)$

(γ) $\quad \forall(\alpha, f),\, (\beta, g) \in A_x\, \big(\beta < \alpha \longrightarrow \exists f' \in \mathcal{F}_{\alpha\kappa}\, \exists k \in \mathcal{F}_{\beta\alpha}\, (g = f' \cdot k \;\;\&\;\; (\alpha, f') \in A_x)\big)$

(δ) $\quad \exists(\alpha, f) \in A_{q\restriction\mathcal{N}}\, \big(\alpha < \alpha^* \;\;\&\;\; \exists h \in \mathcal{F}_{\alpha\alpha^*}\; f = h^* \cdot k \;\;\&\;\; a_{q\restriction\mathcal{N}} \subseteq \mathrm{rge}(f) \;\;\&$

$\qquad\qquad \forall(\beta, g) \in A_{q\restriction\mathcal{N}} \setminus \{(\alpha, f)\}\, (\beta < \alpha \;\;\&\;\; \exists k \in \mathcal{F}_{\beta\alpha}\; g = f \cdot k)\big).$

Finally let $\phi'(x)$ be the conjunction of $\phi(x)$ with

(7) $\qquad \forall i < \lambda\, \big(\nu_i,\, \eta_i \in \mathrm{rge}(h^*) \;\;\&\;\; \exists \zeta_i \in (\nu_i, \eta_i) \cap \mathrm{rge}(h^*)$

$\qquad\qquad\qquad \big(\text{there is a good model } \mathcal{M} \text{ such that } \mathrm{otp}([\nu_i, \zeta_i) \cap \mathrm{rge}(h^*)) \notin \mathcal{M}\big)\big).$

(7) will not be used in the remainder of this section in dealing with the properties of c_{A_q}. It is included only as a gesture towards completeness as it proved useful in [K00] when dealing with local distance functions.

Lemma 8. $H_{\kappa^{++}} \models \phi(q)$ and $H_{\kappa^{++}} \models \phi'(q)$.

Proof. This is easy to see using (δ, F) as the witness (α^*, h^*) for (iii) in the definition of ϕ.

If one could think of more things one could say (or assume and say) about F with parameters from $\mathcal{N}$ these could be added to the above list. (Perhaps the following examination of what can be deduced from the properties (1)-(6) will help in the search to formulate further useful properties.)

($\star$) Now suppose, by the elementarity of $\mathcal{N}$ in $H_{\kappa^{++}}$, that $s \in \mathcal{N}$ and $\mathcal{N} \models \phi(s)$. Let $(\alpha^*, h^*) \in \mathcal{N}$ be the witness to (iii) in the definition of ϕ for s in $\mathcal{N}$.

Of course in the skeletal case of forcing with $\mathbb{P}_0$ itself one does not need any further properties in order to be able to amalgamate such conditions s and q (and hence show that $\mathbb{P}_0$ is $\mathbb{M}$-proper), but it is useful when dealing with forcings which are more fleshed out to have considered properties relating the realms of s and q for which one can find conditions s by reflection.

($\star$) So, set $a_r = a_s \cup a_q$ and $A_r = A_s \cup A_q$.

The intended application of this is that in order to finish the proof of the compatability of q and s one will prove that some $r = (a_r, A_r, \ldots)$ is an element of $\mathbb{P}$ and is stronger than each of q and s.

Key notational definition. Write $c^q(\zeta, \xi)$ in place of $c_{A_q}(\zeta, \xi)$, and similarly for other conditions and putative conditions in $\mathbb{P}$, in order to save on subscripts! So c^s is c_{A_s} and c^r is c_{A_r}. Note that c_r depends only on A_r and not, for example, on having sucessfully shown that $r \in \mathbb{P}$.

I derive a series of extremely useful general facts concerning a_q, a_s, a_r, c^q, c^s and c^r. In order to complete the proof for specific κ-$\mathbb{M}$-proper forcings one needs to use these facts in conjunction with the specifics concealed in the '$\ldots$' part of the working part of conditions.

In particular, Corollary (24) below is a strengthening of (the conclusion of) the Δ-properties of ρ and c. See [M*7] for more on these properties.

Lemma 9. $\forall \zeta_0 < \zeta_1 < \kappa^+\, \big(\zeta_0,\, \zeta_1 \in a_s \longrightarrow c^s(\zeta_0, \zeta_1) < \delta\big)$.

Proof. By Remark (6.i) and clause (7.iii.6) in the definition of ϕ.

Lemma 10. $\forall \xi < \kappa^+ \; \forall (\gamma, g) \in A_q \; \big((\gamma < \delta \;\; \& \;\; \xi \in \mathrm{rge}(g) \;\; \& \;\; \xi \in \mathcal{N}) \longrightarrow \big(\mathrm{rge}(g) \cap \xi \subseteq \mathcal{N} \;\; \& \;\; \xi \in \mathrm{rge}(h^*) \big) \big).$

Proof. Suppose that ξ and (γ, g) are as in the antecedent of the lemma. Thus $\xi \in \mathcal{N} = \mathrm{rge}(F)$. As $\gamma < \delta$ there are some $(\delta, g'') \in \mathcal{F}$ and $g' \in \mathcal{F}_{\gamma\delta}$ such that $g = g'' \cdot g'$. By Fact (I.3), $g'' \restriction \xi_\delta = F \restriction \xi_\delta$, so $\mathrm{rge}(g) \cap \xi \subseteq \mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$. Also as $\gamma < \delta$ there is some $i < \chi$ such that $(\gamma, g) = (\beta_i, f_i)$. As $\mathcal{N} \cap \kappa^+ = \mathrm{rge}(F)$, by the definition of Y_i one has that $\xi \in Y_i$. By (7.iii.2) for s there is some $(\beta_i, f) \in A_s$ such that $Y_i = \mathrm{rge}(f) \cap \mathrm{rge}(h^*)$, so $\xi \in \mathrm{rge}(f) \cap \mathrm{rge}(h^*)$ and $f \restriction \xi_\gamma = g \restriction \xi_\gamma$.

Lemma 11. $\forall \zeta_0 < \zeta_1 < \kappa^+ \big((\zeta_1 \in \mathcal{N} \;\; \& \;\; c^r(\zeta_0, \zeta_1) < \delta) \longrightarrow \big(\zeta_0 \in \mathcal{N} \;\; \& \;\; c^s(\zeta_0, \zeta_1) = c^r(\zeta_0, \zeta_1) \big) \big).$

Proof. Suppose $\zeta_0 < \zeta_1 < \kappa^+$, $\beta < \delta$, and $\zeta_1 \in \mathcal{N}$ and that $\langle \xi_i \,|\, i \leq k \rangle$ and $\langle (\gamma_i, g_i) \,|\, i < k \rangle$ are witnesses that $c^r(\zeta_0, \zeta_1) = \alpha \leq \beta$. Now, $\xi_0 (= \zeta_1) \in \mathcal{N}$, and if $\xi_i \in \mathcal{N}$ then either $(\gamma_i, g_i) \in A_s$, when $\xi_{i+1} \in \mathrm{rge}(g_i) \subseteq \mathcal{N}$, or $(\gamma_i, g_i) \in A_q \setminus A_s$, when by the definition of a witnessing sequence of maps and Lemma (10) one has that $\xi_{i+1} \in \mathrm{rge}(g_i) \subseteq \mathcal{N}$ and $g_i \restriction (\xi_i)_{\gamma_i} = f \restriction (\xi_i)_{\gamma_i}$ for some $(\beta_i, f) \in A_s$. So one has that each $\xi_i \in \mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$ and may replace each $(\gamma_i, g_i) \in A_q \setminus A_s$ by some $(\gamma_i, f) \in A_s$. Thus $\zeta_0 \in \mathcal{N}$ and $c^r(\zeta_0, \zeta_1) = c^s(\zeta_0, \zeta_1) \leq \beta$.

Corollary 12. $\forall \zeta_0 < \zeta_1 < \kappa^+ \; \big(\zeta_0, \; \zeta_1 \in a_s \longrightarrow c^s(\zeta_0, \zeta_1) = c^r(\zeta_0, \zeta_1) < \delta \big).$

Corollary 13. $\forall \xi < \zeta < \kappa^+ \; \big((\xi \notin \mathcal{N} \;\; \& \;\; \zeta \in a_s) \longrightarrow \delta \leq c^r(\xi, \zeta) \big).$

Lemma 14. $e_s \cap e_q = e_q \cap \mathcal{N} \subseteq \beta^\dagger < \alpha^*$.

Proof. $e_s \subseteq \delta$, $e_q \setminus (e_q \cap \mathcal{N}) \cap \delta = \emptyset$, as $\mathcal{N} \cap \kappa = \delta$, and $e_q \cap \mathcal{N} \subseteq e_s$, so $e_s \cap b_q = b_q \cap \mathcal{N}$. For the subset inclusion use (7.iii.5).

Lemma 15. $\forall \nu < \xi < \kappa^+ \; \big(\xi \in a_q \;\; \& \;\; \beta < \alpha^* \longrightarrow (c^r(\nu, \xi) = \beta \longleftrightarrow c^q(\nu, \xi) = \beta) \big).$

Proof. (Similar to the proof of Lemma (11).) Let ν and ξ be as in the antecedent of the lemma. Suppose $\langle \xi_i \,|\, i \leq k \rangle$ and $\langle (\gamma_i, g_i) \,|\, i < k \rangle$ are good witnesses that $c^r(\nu, \xi) = \beta < \alpha^*$. If there is no i such that $\xi_i \in \mathcal{N}$ then, by Remark (6.i), there is no i such that $(\gamma_i, g_i) \in A_s$, so $c^q(\nu, \xi) = \beta$. If there is some i such that $\xi_i \in \mathcal{N}$ then let i^* be the least such. As $c^r(\xi_i, \xi_{i^*}) \leq \gamma_{i-1} < \beta < \alpha^*$ for all $i \in (i^*, k]$, Lemma (11) shows that one then has that $\xi_i \in \mathcal{N}$ for all $i \in [i^*, k]$ and $c^r(\nu, \xi_{i^*}) = c^s(\nu, \xi_{i^*})$. By (7.iii.5) one has that $c^s(\nu, \xi_{i^*}) < \beta^*$. But then, by (7.iii.2), each $(\gamma_i, g_i) \in A_s$, for $i \in [i^*, k)$, can be replaced by some $(\gamma_i, g'_i) \in A_q$. Thus, again, $c^q(\nu, \xi) = \alpha$.

Lemma 16. $\forall \xi_0 < \xi_1 < \kappa^+, \; \forall \beta \in [\delta, \kappa) \; \big(c^q(\xi_0, \xi_1) \leq \beta \longleftrightarrow c^r(\xi_0, \xi_1) \leq \beta \big).$

Proof. Let $\xi_0 < \xi_1 < \kappa^+$, $\beta \in [\delta, \kappa)$ and suppose that $c^r(\xi_0, \xi_1) \leq \beta$. Simply replace any $(\gamma, f) \in A_s$ in a witnessing sequence for this by (δ, F) $(\in A_q)$. The converse is immediate from Lemma (1.3).

Corollary 17. $\forall \xi < \zeta < \kappa^+ \; \big(\xi \in a_q \setminus a_s \;\; \& \;\; \zeta \in a_s \longrightarrow \delta \leq c^r(\xi, \zeta) = c^q(\xi, \zeta) \big).$

Proof. Immediate from Corollary (13) and Lemmas (16).

Lemma 18. Suppose $\zeta \in a_s \setminus a_q$, $\xi \in a_q$, $\zeta < \xi$ and $c^r(\zeta, \xi) < \delta$. Suppose also that $\langle \xi_i \,|\, i \leq k \rangle$

and $\langle(\gamma_i, g_i) \mid i < k\rangle$ are witnesses that $c^r(\zeta, \xi) = \alpha < \delta$. Then there is some $i < k$ such that $\xi_i \in \mathrm{rge}(h^*)$.

Proof. If $\xi \in \mathrm{rge}(h^*)$ there is nothing more to prove. Otherwise let $i < k$ be least such that $\xi_{i+1} \in \mathcal{N}$. As $\xi_i \notin \mathcal{N}$ one has that $(\gamma_i, g_i) \notin \mathcal{N}$ and hence $(\gamma_i, g_i) \in A_q$. Thus $\xi_{i+1} \in \mathrm{rge}(g_i) \cap \mathcal{N}$, and so $\xi_{i+1} \in \mathrm{rge}(h^*)$ by Lemma (10). Finally, $i < k$ since $\xi_k = \zeta \notin \mathrm{rge}(h^*)$, by (7.iii.4).

Lemma 19. $\forall \zeta < \xi < \kappa^+ \left((\zeta \in a_s \setminus a_q \ \ \& \ \ \xi \in a_q) \longrightarrow \alpha^* < c^r(\zeta, \xi)\right)$.

Proof. Let ζ and ξ be as in the antecedent of the lemma. Suppose that $\langle \xi_i \mid i \leq k\rangle$ and $\langle(\gamma_i, g_i) \mid i < k\rangle$ are good witnesses that $c^r(\zeta, \xi) = \alpha$ and that $\alpha \leq \alpha^*$. As $(a_s \setminus a_q) \cap \mathrm{rge}(h^*) = \emptyset$, by (7.iii.4) for s, it suffices to show that $\zeta \in \mathrm{rge}(h^*)$ to obtain a contradiction.

If $\alpha < \alpha^*$ then $c^r(\zeta, \xi) = \alpha = c^q(\zeta, \xi)$ by Lemma (15), so one may as well assume that $(\gamma_i, g_i) \in A_q$ for all $i < k$. But $\zeta \in \mathrm{rge}(g_{k-1})$ and $\zeta \in \mathcal{N}$, so $\zeta \in \mathrm{rge}(h^*)$ by Lemma (10).

If $\alpha = \alpha^*$, applying Lemma (18), let i^* be minimal such that $\xi_{i^*} \in \mathrm{rge}(h^*)$. As $c^r(\zeta, \xi_{i^*}) \leq \alpha$, Lemma (1.4) gives that $\zeta \in \mathrm{rge}(h^*)$.

Lemma 20. $\forall \xi_0 < \xi_1 < \kappa^+ \left(\xi_0, \xi_1 \in a_q \setminus a_s \longrightarrow (c^r(\xi_0, \xi_1) < \delta \longleftrightarrow c^r(\xi_0, \xi_1) < \beta^*)\right)$.

Proof. Let ξ_0 and ξ_1 be as in the antecedent of the lemma. Suppose that $c^r(\xi_0, \xi_1) = \alpha < \delta$ and let $\langle \mu_i \mid i < k\rangle$ and $\langle(\gamma_i, g_i \mid i < k\rangle$ be witnesses to this. By Lemma (11), if one of the $\mu_i \in \mathcal{N}$ then $\xi_0 \in a_q \cap \mathcal{N} = a_q \cap a_s$, a contradiction. So, by Remark (6.i) and the fact that $A_s \subseteq \mathcal{N}$, each $(\gamma_i, g_i) \in A_q \setminus A_s$. Hence $c^q(\xi_0, \xi_1) = \alpha$. But $c^q(\xi_0, \xi_1) < \delta$ if and only if $c^q(\xi_0, \xi_1) < \beta^*$, by the definition of β^*.

Lemma 21. $\forall \tau < \xi < \kappa^+ \left(\tau \in a_s \cap a_q \ \ \& \ \ \xi \in a_q \setminus a_s \longrightarrow (c^r(\tau, \xi) \leq \alpha^* \text{ or } c^r(\tau, \xi) \geq \delta)\right)$.

Proof. Let τ and κ be as in the antecedent of the lemma. Suppose $\langle \mu_i \mid i \leq k\rangle$ and $\langle(\gamma_i, g_i) \mid i < k\rangle$ are good witnesses that $c^r(\tau, \xi) = \beta < \delta$. Then there is a least $i < k$ such that $\mu_{i+1} \in \mathcal{N}$ since $\mu_k \in \mathcal{N}$. As $\mu_j \notin \mathcal{N}$ for $j \leq i$ one has that $(\gamma_j, g_j) \notin \mathcal{N}$, and $\gamma_j < \beta^* < \alpha^*$ since $\gamma_j < \delta$, so $c^r(\mu_{i+1}, \tau) \leq \gamma_i < \alpha^*$. If $k = i+1$ the proof is finished. Otherwise one has that $\mu_{i+1} \in \mathrm{rge}(g_i) \cap \mathcal{N}$, and thus $\mu_{i+1} \in \mathrm{rge}(h^*)$ by Lemma (10). But then $(\alpha^*, h^*) \in A_s$ and $\tau \in a_{q \restriction \mathcal{N}} \subseteq \mathrm{rge}(h^*)$, so τ, $\mu_{i+1} \in \mathrm{rge}(h^*)$. Hence $c^r(\tau, \mu_{i+1}) \leq \alpha^*$, so one has that $c^r(\tau, \xi) \leq \alpha^*$ by the subadditivity of c^r, Theorem (1.12).

Proposition 22. $\forall \zeta < \tau,\ \xi < \kappa^+ \left(\zeta \in a_s \setminus a_q \ \ \& \ \ \tau \in a_s \cap a_q \ \ \& \ \ \xi \in a_q \setminus a_s \longrightarrow c^r(\zeta, \tau) \leq c^r(\zeta, \xi)\right)$.

Proof. Suppose that $c^r(\zeta, \xi) \leq c^r(\zeta, \tau)$. This gives that $\alpha^* < c^r(\zeta, \xi) \leq c^r(\zeta, \tau) < \delta \leq c^r\{\tau, \xi\}$, using Lemma (19), Lemma (9), and Corollary (13) or Lemma (21) for the first, third and fourth inequalities, respectively. Let $\nu = \min(\mathrm{rge}(h^*) \setminus \zeta)$. One has that $\zeta < \nu \leq \min(\{\tau, \xi\})$, the former since $\zeta \notin \mathrm{rge}(h^*)$ as $\mathrm{rge}(h^*) \cap a_s \setminus a_q = \emptyset$ by (7.iii.4) for s, and the latter because $\tau \in \mathrm{rge}(h^*)$, if $\tau < \xi$, and by Lemma (18), which shows that there is some element of $\mathrm{rge}(h^*) \cap [\zeta, \xi]$, if $\xi < \tau$.

As ν, $\tau \in \mathrm{rge}(h^*)$ one has $c^r(\nu, \tau) \leq \alpha^*$. Since $\alpha^* < c^r(\zeta, \tau)$, the subadditivity of c^r gives that $c^r(\zeta, \nu) = c^r(\zeta, \tau)$.

However, if $\langle \xi_i \mid i \leq k\rangle$ and $\langle(\gamma_i, g_i) \mid i < k\rangle$ are good witnesses to the value of $c^r(\zeta, \xi)$ then as noted above, by Lemma (18), there is some $i^* < k-1$ such that $\xi_{i^*} \in \operatorname{rge}(h^*)$. Now, $\langle \gamma_i \mid i < k \rangle$ is strictly increasing, as the witnesses are good, and $c^r(\xi_{i^*}, \xi) \leq \gamma_{i^*-1} < \gamma_{i^*} \leq c^r(\zeta, \xi)$, so $c^r(\zeta, \xi_{i^*}) = c^r(\zeta, \xi)$, by the subadditivity of c^r, again. One also has that $\nu,\ \xi_{i^*} \in \operatorname{rge}(h^*)$, so $c^r(\nu, \xi_{i^*}) \leq \alpha^*$. Thus, using the subadditivity of c^r once more, one has that $c^r(\zeta, \nu) = c^r(\zeta, \xi_{i^*}) = c^r(\zeta, \xi)$.

Taken together these deductions show that $c^r(\zeta, \tau) = c^r(\zeta, \nu) = c^r(\zeta, \xi)$.

Notation 23. If $\nu,\ \mu < \kappa^+$ and $A \in [\mathcal{F}]^{<\omega}$ let $c_A\{\nu, \mu\}$ be $c_A(\nu, \mu)$ if $\nu < \mu$ and be $c_A(\mu, \nu)$ if $\mu < \nu$.

Corollary 24. $\forall \zeta,\ \tau,\ \xi < \kappa^+ \big(\zeta \in a_s \setminus a_q\ \ \&\ \ \tau \in a_s \cap a_q\ \ \&\ \ \xi \in a_q \setminus a_s \longrightarrow c^r\{\zeta, \tau\} \leq c^r\{\zeta, \xi\}\big)$.

Proof. If $\xi < \zeta$ then $c^r\{\tau, \zeta\} < \delta < c^r(\xi, \zeta)$ by Lemma (9) and Corollary (13). If $\zeta < \tau,\ \xi$ then $c^r(\zeta, \tau) \leq c^r(\zeta, \xi)$ by Proposition (22). If $\tau < \zeta < \xi$ and $c^r(\tau, \xi) < \delta$ then $c^r(\tau, \xi) \leq \alpha^*$ by Lemma (21) and $\alpha^* < c^r(\zeta, \xi)$ by Lemma (19), so $c^r(\tau, \zeta) \leq c^r(\zeta, \xi)$ by the subadditivity of c^r. Finally, if $\tau < \zeta < \xi$ and $\delta \leq c^r(\tau, \xi)$ then $c(\tau, \zeta) < \delta \leq c^r(\tau, \xi) \leq c(\zeta, \xi)$ by the subadditivity of c^r, again.

The reader should compare this result with the strengthening of the new Δ-property that was used in [Veličk] and [BS] discussed in the introduction. Corollary (24) gives one a property for c_r which is substantially more powerful again.

Corollary 25. If $\zeta,\ \tau,\ \xi < \kappa^+$, $\zeta \in a_s \setminus a_q$, $\tau \in a_s \cap a_q$ and $\xi \in a_q \setminus a_s$ then

$$\tau,\ \zeta < \xi \longrightarrow \{\gamma \mid c^r\{\gamma, \zeta\} \leq c^r\{\zeta, \tau\}\} \subseteq \{\gamma \mid c^r\{\gamma, \zeta\} \leq c^r\{\zeta, \xi\}\}$$

Corollary 26. Summarising, perhaps repetiously, if $\zeta,\ \tau,\ \xi < \kappa^+$, $\zeta \in a_s \setminus a_q$, $\tau \in a_s \cap a_q$ and $\xi \in a_q \setminus a_s$ then

$$\begin{aligned}
\tau < \zeta < \xi &\longrightarrow c^r(\tau, \zeta),\ c^r(\tau, \xi) \leq c^r(\zeta, \xi),\\
\tau < \xi < \zeta &\longrightarrow c^r(\tau, \zeta),\ c^r(\tau, \xi) \leq c^r(\xi, \zeta),\\
\zeta < \tau < \xi &\longrightarrow c^r(\zeta, \tau) \leq c^r(\zeta, \xi) \leq c(\tau, \xi),\\
\xi < \tau < \zeta &\longrightarrow c^r(\tau, \zeta) < c^r(\xi, \tau) \leq c^r(\xi, \zeta),\\
\zeta < \xi < \tau &\longrightarrow c^r(\zeta, \tau) \leq c^r(\zeta, \xi) \leq c(\xi, \tau),\\
\xi < \zeta < \tau &\longrightarrow c^r(\zeta, \tau) < c^r(\xi, \zeta) = c^r(\xi, \tau).
\end{aligned}$$

Lemma 27. $\forall \xi_0 < \xi_1 < \kappa^+ \big(\xi_0,\ \xi_1 \in a_q \setminus a_s \longrightarrow c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)\big)$.

Proof. Let ξ_0 and ξ_1 be as in the antecedent of the lemma. By Lemma (16), if $c^r(\xi_0, \xi_1) \geq \delta$ then $c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$. By Lemma (20), $c^r(\xi_0, \xi_1) < \delta$ implies $c^r(\xi_0, \xi_1) < \alpha^*$. But Lemma (15) gives that $c^r(\xi_0, \xi_1) < \alpha^*$ implies $c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$.

Lemma 28. $\forall \tau_0 < \tau_1 < \kappa^+ \big(\tau_0,\ \tau_1 \in a_q \cap a_s \longrightarrow c^r(\tau_0, \tau_1) \leq \alpha^*\big)$.

Proof. This is immediate from the fact, already used several times, that $a_q \cap \mathcal{N} \subseteq \operatorname{rge}(h^*)$, which is given by clause (7.iii.3) of the definition of ϕ.

Lemma 29. $\forall \tau_0 < \tau_1 < \tau_2 < \kappa^+ \big(\tau_0,\ \tau_1,\ \tau_2 \in a_q \cap a_s \longrightarrow$

$$\big(c^q(\tau_0, \tau_1) \leq c^q(\tau_1, \tau_2) \longrightarrow c^{q \restriction \mathcal{N}}(\tau_0, \tau_1) \leq c^{q \restriction \mathcal{N}}(\tau_1, \tau_2)\big)\big).$$

Proof. This is immediate from Lemma (1.11).

Lemma 30. If $\xi \in a_q$, $\tau \in a_q \cap a_s$, $\xi < \tau$ and $c^r(\xi, \tau) \le \delta$ then $\xi \in \mathrm{rge}(F)$, so $\xi \in a_q \cap a_s$ and $c^r(\xi, \tau) \le \alpha^*$.

Proof. It is immediate that $\xi \in \mathrm{rge}(F)$ by Lemma (1.4). The remainder of the lemma follows from the fact that $a_s \cap a_q = a_q \cap \mathcal{N}$ and Lemma (28).

I give a couple of final corollaries of the work of this section showing some circumstances when the hypotheses of Corollaries (3.9) and (3.10) are satisfied.

Corollary 31. Suppose that $\nu < \tau < \kappa^+$. If $\tau \in \mathcal{N}$, $\alpha < \delta$, $A = A_s$ and $B = A_r$, or if $\tau \in a_q$ (or even merely $\tau \in (\kappa^+ \setminus \mathcal{N}) \cup \mathrm{rge}(h^*)$), $\alpha < \alpha^*$, $A = A_q$ and $B = A_r$, or if $\alpha \in [\delta, \kappa)$, $A = A_q$ and $B = A_r$, then the hypotheses of Corollary (3.9) are satisfied.

Proof. This is shown in the proofs of Lemmas (11), (15) and (16), respectively.

Corollary 32. If ν, $\tau \in \kappa^+ \setminus \mathcal{N}$, $\beta^* \le \beta$ and $\alpha < \delta$ then the hypotheses of Corollary (3.10) are satisfied.

Proof. This is immediate from Lemma (10).

This concludes the discussion of the relationship between c^q, c^s and c^r based on the fact that $\mathcal{N} \models \phi(s)$ and using the analysis of c_A given in §1.

§5. Two applications.

Scattered spaces of size μ^{++}.

Recall the following topological definitions.

Definition 1. A topological space X is *μ-Lindelöf* if every open cover has a subcover of size *less than* μ. Thus ω-Lindeöf means compact and ω_1-Lindelöf means Lindelöf. X is locally μ-Lindelöf if every point has an open neighbourhood whose closure is μ-Lindelof. A space is 0-dimensional if its topology has a basis of sets which are both closed and open. A space X is *scattered* or *right-separated* if there is a well ordering $<_X$ of it such that for every $x \in X$ there is an open neighbourhood $\mathcal{O}$ of x such that $\mathcal{O} \cap \{y \in X \mid x <_X y\} = \emptyset$. A space X is μ-tight if for every $Y \subseteq X$ and x is in the closure of Y there is some $Z \in [Y]^{\le\mu}$ such that x is in the closure of Z. The *Cantor-Bendixson derivative* of a space is the space obtained by discarding all isolated points with the relative topology. The *height* of a scattered space is its Cantor-Bendixson rank – the ordinal number of times one can iterate taking the Cantor-Bendixson derivative of a space (taking intersections at limit stages of the iteration). The *width* of a scattered space is the supremum of the size of the sets discarded at each stage when iterating taking the Cantor-Bendixson derivative.

Theorem 2. If there is a stationary $(\kappa, 1)$-simplified morass and κ is the successor of a regular cardinal μ with $\kappa^{+<\mu} = \kappa^+$ then there is a cardinal preserving forcing which adds a 0-dimensional, locally μ-Lindelöf, μ-tight, scattered space of separbility degree μ, size and height κ^+ and width μ. The one-point μ-Lindelöfization of the space has the same properties, but is μ-Lindelöf rather than locally μ-Lindelöf.

A space as described in Theorem (2) was added for $\mu = \omega$ by [BS] — although they worked in terms of superatomic Boolean algebras, Stone duality gives an immediate translation. It would be nice to be able to prove that one can force to add a 0-dimensional, locally compact, countably tight, scattered space of size μ^{++}, however it seems unlikely that one will be able to do this with conditions whose working parts are not finite. I sketch why this is so at the end of the proof of Theorem (2).

Definition 3. Define a partial order $\mathbb{P}^-$ with elements of the form $p = (a_p, \leq_p, i_p)$, where $a_p \in [\kappa^+]^{<\mu}$, $\leq_p$ is a partial order on a_p such that $\alpha \leq_p \beta$ implies $\alpha \leq \beta$, and $i_p : [a_p]^2 \longrightarrow [a_p]^{<\mu}$ is defined by $i_p(\alpha, \beta) = \{\gamma \in a_p \,|\, \gamma \leq \alpha,\ \beta \ \ \&\ \ \forall \varepsilon \in a_p\ (\neg(\gamma <_p \varepsilon \leq_p \alpha,\ \beta))\}$ and is included in the condition p purely for notational purposes. (Here $\gamma <_p \varepsilon$ abbreviates $\gamma \leq_p \varepsilon \ \ \&\ \ \gamma \neq \varepsilon$, as usual). The order on $\mathbb{P}^-$ is $p' \leq p$ if $a_p \subseteq a_{p'}$, $\leq_p = \leq_{p'} \restriction a_p \times a_p$ and $i_p = i_{p'} \restriction [a_p]^2$.

Of course one may equally well write $p = (a_p, \leq_p, i_p)$ as $p = (a_p, h_p, i_p)$ where $h_p : a_p \times a_p \longrightarrow 2$ is a function such that $\forall \alpha \in a_p\ h_p(\alpha, \alpha) = 1$; $\forall \alpha,\ \beta \in a_p\ (\alpha < \beta \longrightarrow h_p(\alpha, \beta) = 0$; and $\forall \alpha,\ \beta,\ \gamma \in a_p\ (h_p(\alpha, \beta) = 1 \ \ \&\ \ h_p(\beta, \gamma) = 1 \longrightarrow h_p(\alpha, \gamma) = 1)$, that is, that h_p satisfies the requirements to be the characteristic function of a partial order on a_p.

Note that $\mathbb{P}^-$ does not have the κ-chain condition: if one takes κ many conditions and applies the Δ-system lemma to their 'a's then it may still be the case that no two of the 'i's agree on their restriction to the root of the Δ-system.

Definition 4. Define a forcing notion $\mathbb{P}$ with conditions $p = (a_p, \leq_p, i_p, A_p)$ where $(a_p, \leq_p, i_p) \in \mathbb{P}^-$, $A_p \in \mathcal{F}$ and, writing c^p for c_{A_p},

$$\forall \alpha,\ \beta \in a_p\ (\alpha < \beta \longrightarrow i_p(\alpha, \beta) \subseteq \{\gamma \leq \alpha \,|\, c^p(\gamma, \alpha) \leq c^p(\alpha, \beta)\}).$$

The order is the product of the order on $\mathbb{P}^-$ and reverse inclusion, *i.e.* $p' \leq p$ if $a_p \subseteq a_{p'}$, $\leq_p = \leq_{p'} \restriction a_p \times a_p$, $i_p = i_{p'} \restriction [a_p]^2$ and $A_{p'} = A_p$. a_p is the *realm* of p and A_p is the *side condition part* of p.

The intention is that if G is $\mathbb{P}$-generic one will take $B_\alpha = \{\beta \leq \alpha \,|\, \exists p \in G\ (\beta,\ \alpha \in a_p \ \ \&\ \ \beta \leq_p \alpha)\}$. Taking $\mathcal{B} = \{B_\alpha \setminus \bigcup\{B_\beta \,|\, \beta \in b\} \,|\, \alpha < \kappa^+ \ \ \&\ \ b \in [\alpha]^{<\mu}\}$ will give a basis for a topology on κ^+ which satisfies the theorem provided that cardinals are preserved. I prove the topological properties after showing that cardinals are indeed preserved by the forcing.

Proposition 5. $\mathbb{P}$ is κ-$\mathbb{M}$-proper.

Proof. So suppose that $p \in \mathbb{P}$ and that $\mathcal{N}$ is as in the general framework of §4 but for this specific $\mathbb{P}$.

Step A. Checking that $p' = (a_p, \leq_p, i_p, A_p \cup \{(\delta, F)\})$ is a condition and is stronger than p.

Proof. Observe that $(\alpha, f) \in \mathcal{N}$ for each $(\alpha, f) \in A_p$, since $p \subseteq \mathcal{N}$. So for each $(\alpha, f) \in A_p$ one has, by (i) of the properties of $\mathcal{N}$ given in §2, that there is some $f' \in \mathcal{F}_{\alpha\beta}$ such that $f = F \cdot f'$. Suppose that α, β, $\gamma \in a_p$, $\gamma < \alpha < \beta$ and $c^p(\gamma,\alpha) \leq c^p(\alpha,\beta)$. As $a_p \subseteq \mathcal{N} \cap \kappa^+ = \mathrm{rge}(F)$ one has that $c^{p'}(\gamma,\alpha) \leq c^{p'}(\alpha,\beta)$ by Lemma (1.6). This shows that $i_{p'}(\alpha,\beta) \subseteq \{\gamma < \alpha \,|\, c^{p'}(\gamma,\alpha) \leq c^{p'}(\alpha,\beta)\}$. Hence $p' \in \mathbb{P}$.

Suppose that $\mathcal{D}$ is a dense, open subset of $\mathbb{P}$, $\mathcal{D} \in \mathcal{N}$ and $q \in \mathcal{D}$ where $q \leq p'$.

Step B. Show that without loss of generality one may assume that $c^q(\nu,\tau) < \kappa$ for all ν, $\tau \in a_q$.

Proof. For if q does not satisfy this one may simply add to A_q a map (β, f) such that $a_q \subseteq \mathrm{rge}(f)$ and $\alpha \leq \beta$ for all $(\alpha, g) \in A_q$ by applying Fact (I.5). By Lemma (1.6), this augmentation of q is still a condition, it is stronger than q by the definition of $\leq_{\mathbb{P}}$, and is in $\mathcal{D}$ because $\mathcal{D}$ is open.

Let $q \restriction \mathcal{N} = q \cap \mathcal{N}$.

Step C. Show that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and $q \leq q \restriction \mathcal{N}$.

Lemma 6. If τ_0, $\tau_1 \in a_q \cap \mathcal{N}$ then $i_q(\tau_0,\tau_1) \subseteq \mathcal{N}$. And if $\gamma < \tau_0 < \tau_1$ and $c^q(\gamma,\tau_0) \leq c^q(\tau_0,\tau_1)$ then $c^{q\restriction\mathcal{N}}(\gamma,\tau_0) \leq c^{q\restriction\mathcal{N}}(\tau_0,\tau_1)$.

Proof. As τ_0, $\tau_1 \in \mathcal{N} = \mathrm{rge}(F)$, one has that $c^q(\tau_0,\tau_1) \leq \delta$. If $\gamma \in i_q(\tau_0,\tau_1)$ then $c^q(\gamma,\tau_0) \leq \delta$ by the constraint on i_q in the definition of what it is to be a condition in $\mathbb{P}$. So by Lemma (1.4) one has that $\gamma \in \mathrm{rge}(F)$. By Lemma (1.11), for every $\gamma < \tau_0$, if $c^q(\gamma,\tau_0) \leq c^q(\tau_0,\tau_1)$ one has that $c^{q\restriction\mathcal{N}}(\gamma,\tau_0) \leq c^{q\restriction\mathcal{N}}(\tau_0,\tau_1)$.

Corollary 7. $q \leq q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$.

Proof. If τ_0, $\tau_1 \in a_q \cap \mathcal{N}$ and $\gamma \in i_q(\tau_0,\tau_1)$ then, by the first half of Lemma (6), $\gamma \in a_{q\restriction N}$ and so $i_{q\restriction\mathcal{N}}(\tau_0,\tau_1) = i_q(\tau_0,\tau_1)$. So $i_{q\restriction\mathcal{N}}(\tau_0,\tau_1) = i_q(\tau_0,\tau_1) \subseteq \{\gamma \leq \tau_0 \,|\, c^q(\gamma,\tau_0) \leq c^q(\gamma,\tau_0)\} \subseteq \{\gamma \leq \tau_0 \,|\, c^{q\restriction\mathcal{N}}(\gamma,\tau_0) \leq c^{q\restriction\mathcal{N}}(\gamma,\tau_0)\}$, by the second half of Lemma (6). Hence as far as the pair τ_0, τ_1 are concerned, $q \restriction \mathcal{N}$ satisfies the requirement for being a condition, and q and $q \restriction \mathcal{N}$ satisfy the requirement for q to be a stronger condition than $q \restriction \mathcal{N}$.

Now obtain s as in §2 using b_q as the arbitrary set e_q in the section of Notation in §2.

Step D. Define some r with realm $a_s \cup a_q$ and side condition part $A_s \cup A_q$. Show that $r \in \mathbb{P}$ and that $r \leq s$, q.

Proof. I define r as follows. Let $a_r = a_s \cup a_q$ and $A_r = A_s \cup A_q$. Let $\alpha \leq_r \beta$ if $\alpha \leq_q \beta$ for $\alpha \leq_s \beta$ or $\exists\gamma\ (\alpha \leq_s \gamma \leq_q \beta)$ or $\exists\gamma\ (\alpha \leq_q \gamma \leq_s \beta)$.

Lemma 8. If $\{\alpha,\gamma\} \in [a_q]^2$, $\beta \in a_s$ and ε, $\nu \in a_s \cap a_q$ are such that $\alpha \leq_q \varepsilon \leq_s \beta \leq_s \nu \leq_q \gamma$ then $\alpha \leq_q \gamma$. The same conclusion also holds with s and q exchanged.

Proof. One has that $\alpha \leq_q \varepsilon \leq_s \nu \leq_q \gamma$ by the transitivity of $\leq_s$. Moreover, $\varepsilon \leq_q \nu$ since $\leq_q\restriction a_{q\restriction\mathcal{N}} = \leq_s\restriction a_{q\restriction\mathcal{N}}$, so $\alpha \leq_q \gamma$ by the transitivity of $\leq_q$. The proof with q and s exchanged is identical.

Corollary 9. $\leq_r \upharpoonright a_q \times a_q = \leq_q$ and $\leq_r \upharpoonright a_s \times a_s = \leq_s$.

Lemma 10. $i_r \upharpoonright [a_q]^2 = i_q$ and $i_r \upharpoonright [a_s]^2 = i_s$.

Proof. Suppose ξ_0, $\xi_1 \in a_q$ and $\nu \leq_r \xi_0, \xi_1$. If $\nu \in a_q$ then $\nu \leq_q \xi_0, \xi_1$, by Corollary (9), so there is some $\xi \in i_q(\xi_0, \xi_1)$ such that $\nu \leq_q \xi$. If $\nu \in a_s \setminus a_q$ then there are τ_0, $\tau_1 \in a_s \cap a_q$ such that $\nu \leq_s \tau_0 \leq_q \xi_0$ and $\nu \leq_s \tau_1 \leq_q \xi_1$. Let $\tau \in i_s(\tau_0, \tau_1)$ be such that $\nu \leq_s \tau$. Now, as q, $s \leq q \upharpoonright \mathcal{N}$ one has that $i_q(\tau_0, \tau_1) = i_{q\upharpoonright\mathcal{N}}(\tau_0, \tau_1) = i_s(\tau_0, \tau_1)$. So $\tau \in i_q(\tau_0, \tau_1)$. Let $\xi \in i_q(\xi_0, \xi_1)$ be such that $\tau \leq_q \xi$. Then $\nu \leq_r \tau \leq_r \xi \leq_r \xi_0, \xi_1$, so $\nu \leq_r \xi \in i_q(\xi_0, \xi_1)$. This shows that $i_r \upharpoonright [a_q]^2 = i_q$. The argument that $i_r \upharpoonright [a_s]^2 = i_s$ is identical with the rôles of s and q swapped.

Lemma 11. $\leq_r$ is a partial order on $a_r \times a_r$.

Proof. $\leq_r$ is clearly reflexive and anti-symmetric, so only transitiviy needs to be shown. Suppose $\alpha \leq_r \beta \leq_r \gamma$. If $\{\alpha, \beta, \gamma\} \in [a_q]^3 \cup [a_s]^3$ then $\alpha \leq_r \gamma$. If $\{\alpha, \gamma\} \in [a_q]^2$ and $\beta \in a_s$ then $\alpha \leq_q \gamma$ by Lemma (8). If $\{\alpha, \beta\} \in [a_q]^2$ and $\gamma \in a_s$ let $\varepsilon \in a_s \cap a_q$ be such that $\alpha \leq_q \beta \leq_q \varepsilon \leq_s \gamma$. Then $\alpha \leq_q \varepsilon \leq_s \gamma$ by the transitivity of $\leq_q$. The argument in each of the other cases is identical to one of these latter two.

The arguments for Lemmas (8), (10) and (11) are essentially the same as analgous arguments used in [BS, §8] in the proof that the subcollection of $\mathbb{P}^-$ (when $\kappa = \omega_1$) consisting of those conditions p with $i_p(\alpha, \beta) \subseteq \{\gamma \leq \alpha \mid c(\gamma, \alpha) \leq c(\alpha, \beta)\}$ for all α, $\beta \in a_p$, under the suborder of $\leq_{\mathbb{P}^-}$ such that that i remains fixed between a condition and a stronger one, has the ccc. Similarly to that proof, as a consequence of Lemmas (8), (10) and (11) it suffices to show that $r \in \mathbb{P}$ in order to show that q and s are compatible in $\mathbb{P}$. Here this will show that $\mathbb{P}$ is $\mathbb{M}$-proper. What needs to be done in order to complete this is to show that if α, $\beta \in a_r$ with $\alpha < \beta$ and $\gamma \in i_r(\alpha, \beta)$ then $c^r(\gamma, \alpha) \leq c^r(\alpha, \beta)$. But the proof of this is markedly different from (and harder than) the analgous proof in [BS, §8]. In conformity with the usage in §4, I try to use τ, possibly adorned with subscripts, for variables in $a_s \cap a_q$, ζ for variables in a_s and ξ for variables in a_q.

Proposition 12. $r \in \mathbb{P}$.

Proof.

Case A. ξ_0, $\xi_1 \in a_q$ and $\xi_0 < \xi_1$. Lemma (10) shows that $i_r(\xi_0, \xi_1) = i_q(\xi_0, \xi_1)$. So what needs to be shown is that if $\xi \in i_q(\xi_0, \xi_1)$ then $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$. Suppose that $\xi \in i_q(\xi_0, \xi_1)$.

Subcase A.i. ξ_0, $\xi_1 \in a_q \cap a_s = a_{q\upharpoonright\mathcal{N}}$. If $\xi \in i_q(\xi_0, \xi_1)$ then $\xi \in a_q \cap \mathcal{N} = a_q \cap a_s$, by Lemma (6) and clauses (iii.3) and (iii.4) of the definition of ϕ for q, respectively. Thus $\xi \in i_{q\upharpoonright\mathcal{N}}(\xi_0, \xi_1)$. As $s \leq q \upharpoonright \mathcal{N}$ one has that $\xi \in i_s(\xi_0, \xi_1)$, and hence that $c^s(\xi, \xi_0) \leq c^s(\xi_0, \xi_1)$. By (4.12) one has that $c^s(\xi, \xi_0) = c^r(\xi, \xi_0)$ and $c^s(\xi_0, \xi_1) = c^r(\xi_0, \xi_1)$. Thus $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$.

Subcase A.ii. ξ_0, $\xi_1 \in a_q \setminus a_s$. By Lemma (1.3) one has that $c^r(\xi, \xi_0) \leq c^q(\xi, \xi_0)$, and one also has that $c^q(\xi, \xi_0) \leq c^q(\xi_0, \xi_1)$ since $q \in \mathbb{P}$ and $\xi \in i_q(\xi_0, \xi_1)$. Moreover, $c^q(\xi_0, \xi_1) = c^r(\xi_0, \xi_1)$ by (4.27). Hence $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$.

Subcase A.iii. $\xi_0 \in a_q \setminus a_s$ and $\xi_1 \in a_q \cap a_s = a_{q\upharpoonright\mathcal{N}}$. One has $c^r(\xi, \xi_0) \leq c^q(\xi, \xi_0)$ by Lemma (1.3), again, and $c^q(\xi, \xi_0) \leq c^q(\xi_0, \xi_1)$ since $q \in \mathbb{P}$ and $\xi \in i_q(\xi_0, \xi_1)$. One also has that

$\delta \leq c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$ by (4.17). Hence $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$.

Subcase A.iv. $\xi_0 \in a_q \cap a_s$ and $\xi_1 \in a_q \setminus a_s$. If $\delta \leq c^r(\xi_0, \xi_1)$ then $c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$ by (4.16), when $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$ as in Cases (A.ii) and (A.iii) again. Otherwise, $c^r(\xi_0, \xi_1) \leq \alpha^*$ by (4.21). If $c^r(\xi_0, \xi_1) < \alpha^*$ then $c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$ by (4.15), and $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$ as in Cases (A.ii) and (A.iii) once more. The remaining possibility is that $c^r(\xi_0, \xi_1) = \alpha^*$ and $c^q(\xi_0, \xi_1) = \delta$. As $\xi \in a_q$ one has that $c^q(\xi, \xi_0) \leq c^q(\xi, \xi_1) = \delta$. By Lemma (1.4) one has that $\xi \in \mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$, so that $\xi \in a_q \cap a_s$. By clause (iii.3) of the definition of ϕ for s one then has that ξ, $\xi_0 \in \mathrm{rge}(h^*)$, so that $c^r(\xi, \xi_0) \leq \alpha^*$, as required.

Case B. ζ_0, $\zeta_1 \in a_s$ and $\zeta_0 < \zeta_1$. Lemma (10) shows that $i_r(\zeta_0, \zeta_1) = i_s(\zeta_0, \zeta_1)$. So what needs to be shown is that if $\zeta \in i_s(\zeta_0, \zeta_1)$ then $c^r(\zeta, \zeta_0) \leq c^r(\zeta_0, \zeta_1)$. So suppose that $\zeta \in i_q(\zeta_0, \zeta_1)$. As previously, one has that $c^r(\zeta, \zeta_0) \leq c^s(\zeta, \zeta_0)$ by Lemma (1.3), and that $c^s(\zeta, \zeta_0) \leq c^s(\zeta_0, \zeta_1)$ since $s \in \mathbb{P}$ and $\zeta \in i_s(\zeta_0, \zeta_1)$. But (4.12) shows that $c^r(\zeta_0, \zeta_1) = c^s(\zeta_0, \zeta_1)$. So $c^r(\zeta, \zeta_0) \leq c^r(\zeta_0, \zeta_1)$ as required.

Case C. $\zeta \in a_s \setminus a_q$ and $\xi \in a_q \setminus a_s$. If $\zeta \leq_r \xi$ then $i_r(\zeta, \xi) = \{\zeta\}$ and, clearly, $c^r(\zeta, \zeta) \leq c^r(\zeta, \xi)$. Similarly if $\xi \leq_r \zeta$ then $i_r(\xi, \zeta) = \{\xi\}$ and $c^r(\xi, \xi) \leq c^r(\xi, \zeta)$. So suppose that $\zeta \not\leq_r \xi$ and $\xi \not\leq_r \zeta$.

Let $\nu \in i_r\{\xi, \zeta\}$. In order to complete the proof one has to show that $c^r(\nu, \zeta) \leq c^r\{\zeta, \xi\}$.

Subcase C.i. $\nu \in a_s \cap a_q$. By (4.24) one immediately has $c^r(\nu, \zeta) \leq c^r\{\zeta, \xi\}$.

Subcase C.ii. $\nu \in a_s \setminus a^q$.

By the definition of $\leq_r$, let $\tau \in a_s \cap a_q$ be such that $\nu \leq_s \tau \leq_q \xi$. Clearly $\nu \in i_r\{\tau, \zeta\}$, and so by Case (B) above one has that $c^r(\nu, \tau)$, $c^r(\nu, \zeta) \leq c^r\{\tau, \zeta\}$. However, by (4.24) again, one has that $c^r\{\tau, \zeta\} \leq c^r\{\zeta, \xi\}$. So $c^r(\nu, \zeta) \leq c^r\{\zeta, \xi\}$, and by the subadditivity of c^r one has that $c^r(\nu, \zeta)$, $c^r(\nu, \xi) \leq c^r\{\zeta, \xi\}$.

Subcase C.iii. $\nu \in a_q \setminus a_s$. Exactly as Subcase (C.ii) but switching the rôles of s, ζ and q, ξ, and using Case (A) in place of Case (B).

Hence Cases (A), (B) and (C) taken together show that if α, $\beta \in a_r$ with $\alpha < \beta$ and $\gamma \in i_r(\alpha, \beta)$ then $c^r(\gamma, \alpha) \leq c^r(\alpha, \beta)$, and so that $r \in \mathbb{P}$, thus concluding the proof Proposition (12).

This shows that q and s are compatible and hence $\mathbb{P}$ is κ-$\mathbb{M}$-proper, concluding the proof of Proposition (5). Hence forcing with $\mathbb{P}$ preserves κ.

One great advantage of κ-$\mathbb{M}$-proper forcing over proper forcing is that one has a reasonable chance of preserving κ^+ (and all greater cardinals) as well as κ!

Proposition 13. $\mathbb{P}$ has the κ^+-cc.

Proof. Let $\{p'_\nu \mid \nu < \kappa^+\}$ be a subset of $\mathbb{P}$ of size κ^+. Extend each p'_ν in the original collection one is given to a stronger condition p_ν by adding a single map to $A_{p'_\nu}$ if necessary in order to ensure that there is a map $(\alpha_\nu, F_\nu) \in A_{p_\nu}$ such that for all $(\beta, f) \in A_{p_\nu}$ one has $\beta \leq \alpha$ and there is some $g \in \mathcal{F}_{\beta\alpha}$ such that $f = F \cdot g$ and such that $a_p \subseteq \mathrm{rge}(F)$. This is possible by Fact (I.5), as in the argument in the paragraph two prior to Lemma (6) above where q is introduced and then assumed to have this property. If one can find η, $\nu < \delta$ such that p_η and p_ν are compatible then, clearly, one will have that p'_η and p'_ν are compatible as well.

By thinning using the Δ-system lemma if necessary one may as well assume that there is some $a \in [\kappa^+]^{<\mu}$ such that a is an initial segment of each a_{p_ν}. One may as well also assume that if $\eta < \nu < \kappa^+$ then $\max(a_{p_\eta}) < \min(a_{p_\nu} \setminus a)$, and there is an order preserving isomorphism from $(a_{p_\eta}, \leq_{p_\eta})$ to $(a_{p_\nu}, \leq_{p_\nu})$. Moreover, one may as well assume that there is some $\alpha < \kappa$, some $A \in [\bigcup\{\{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha\}]^{<\mu}$ and for each $\nu < \kappa^+$ there is a map $F_\nu \in \mathcal{F}_{\alpha\kappa}$ such that $A_{p_\nu} = \{(\beta, F_\nu \cdot h) \mid (\beta, h) \in A\} \cup \{(\alpha, F_\nu)\}$ and such that $F_\nu \cdot F_\eta^{-1} \restriction a_{p_\eta}$ is the order preserving isomorphism from $(a_{p_\eta}, \leq_{p_\eta})$ to $(a_{p_\nu}, \leq_{p_\nu})$. This is possible using Proposition (I.13).

Clearly for each $\nu < \kappa^+$ and $\xi < \zeta < \kappa^+$ one has that $c^{p_\nu}(\xi, \zeta) \leq \alpha$ if and only if $c^{p_\nu}(\xi, \zeta) < \kappa$. Let $\eta < \nu < \kappa^+$ and set $a_r = a_{p_\eta} \cup a_{p_\eta}$, $\leq_r = \leq_{p_\eta} \cup \leq_{p_\eta}$ and $A_r = A_{p_\eta} \cup A_{p_\eta}$. Note that $i_r(\zeta_0, \zeta_1) = i_{p_\eta}$ if ζ_0, $\zeta_1 \in a_{p_\eta}$ and $i_r(\xi_0, \xi_1) = i_{p_\nu}$ if ξ_0, $\xi_1 \in a_{p_\nu}$. It is also clear that if $\xi < \zeta < \kappa^+$ then $c^r(\xi, \zeta) \leq \alpha$ if and only if $c^r(\xi, \zeta) < \kappa$. By Lemma (1.14), one has that if $\zeta \in \operatorname{rge}(F_\eta)$ then $c^r(\xi, \zeta) = c^{p_\eta}(\xi, \zeta)$ and if $\zeta \in \operatorname{rge}(F_\nu)$ then $c^r(\xi, \zeta) = c^{p_\nu}(\xi, \zeta)$.

Claim 14. If $\xi \in a_{p_\eta} \setminus a_{p_\nu}$ then $\xi \notin \operatorname{rge}(F_\nu)$.

Proof. If $\xi \in \operatorname{rge}(F_\nu)$ then $\xi \in \operatorname{rge}(F_\nu) \cap \operatorname{rge}(F_\eta)$ since $F_\nu \cdot F_\eta^{-1} \restriction a_{p_\eta}$ is the order preserving isomorphism from a_{p_η} to a_{p_ν}. But then one has that $F_\nu(\xi_\alpha) = \xi = F_\eta(\xi_\alpha)$ by Fact (I.3), so $F_\nu \cdot F_\eta^{-1}(\xi) = \xi$ and hence $\xi \in a_{p_\nu} \cap a_{p_\eta}$, a contradiction.

It is now immediate that if $\xi \in a_{p_\eta} \setminus a_{p_\nu}$ and $\zeta \in a_{p_\nu} \setminus a_{p_\eta}$ then $\xi \notin \operatorname{rge}(F_\nu)$, by Claim (14), so that $c^{p_\nu}(\xi, \zeta) = \kappa$, and thus $c^r(\xi, \zeta) = \kappa$ from Claim (14). Hence for such ξ, ζ one has that $i_r(\xi, \zeta) \subseteq \{\tau \leq \xi \mid c^r(\tau, \xi) \leq c^r(\xi, \zeta)\}$.

By Claim (14), as well, if ξ_0, $\xi_1 \in a_{p_\eta}$ then $c^r(\xi_0, \xi_1) = c^{p_\eta}(\xi_0, \xi_1)$; and so if $\xi_0 < \xi_1$ one has that $i_r(\xi_0, \xi_1) \subseteq \{\tau \leq \xi_0 \mid e^{p_\eta}(\tau, \xi_0) \leq e^{p_\eta}(\xi_0, \xi_1)\} = \{\tau \leq \xi_0 \mid e^r(\tau, \xi_0) \leq e^r(\xi_0, \xi_1)\}$. (One gets equality rather than just the necessary inclusion by applying Claim (14) again for $\tau < \xi_0 \in \operatorname{rge}(F_\eta)$.) Similarly, if ζ_0, $\zeta_1 \in a_{p_\nu}$ then $c^r(\zeta_0, \zeta_1) = c^{p_\nu}(\zeta_0, \zeta_1)$. Thus $i_r(\zeta_0, \zeta_1) \subseteq \{\tau \leq \zeta_0 \mid e^r(\tau, \zeta_0) \leq e^r(\zeta_0, \zeta_1)\}$ if $\zeta_0 < \zeta_1$.

Hence $r \in \mathbb{P}$, and thus $r \leq_{\mathbb{P}} p_\eta$, p_ν. So $\mathbb{P}$ does have the κ^+-cc.

Thus cardinals above μ are preserved by forcing with $\mathbb{P}$. The next proposition shows that cardinals up to μ are also preserved.

Proposition 15. $\mathbb{P}$ is μ-closed.

Proof. **Proof.** Let $\langle p_\varepsilon \mid \varepsilon < \chi \rangle$ be a descending sequence of conditions for some limit ordinal $\chi < \mu$. Define p by setting $a_p = \bigcup\{a_{p_\varepsilon} \mid \varepsilon < \chi\}$, $\nu \leq_p \tau$ if there is some $\varepsilon < \chi$ such that $\nu \leq_{p_\varepsilon} \tau$ and $A_p = \bigcup\{A_{p_\varepsilon} \mid \varepsilon < \chi\}$.

First of all, I show that $p \in \mathbb{P}$. Suppose that η, ν, $\tau \in a_p$ and $\eta \in i_p(\nu, \tau)$. Let $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ and $\langle \zeta_j \mid j \leq l \rangle$ and $\langle (\beta_j, f_j) \mid j < l \rangle$ be witnesses to the values of $c^p(\eta, \nu)$ and $c^p(\nu, \tau)$ respectively. Then there is some $\varepsilon < \chi$ such that $\{(\gamma_i, g_i) \mid i < k\} \cup \{(\beta_j, f_j) \mid j < l\} \subseteq A_{p_\varepsilon}$

and η, ν, $\tau \in a_{p_\varepsilon}$. First of all note, by the definition of i_{p_ε} and the fact that $a_{p_\varepsilon} \subseteq a_p$, that $\eta \in i_{p_\varepsilon}(\nu, \tau)$. As p_ε is a condition this implies that $c^{p_\varepsilon}(\eta, \nu) \leq c^{p_\varepsilon}(\nu, \tau)$. But as $c^p(\eta, \nu) \leq c^{p_\varepsilon}(\eta, \nu)$ and $c^p(\nu, \tau) \leq c^{p_\varepsilon}(\nu, \tau)$, by Lemma (1.3), one has that $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ and $\langle \zeta_j \mid j \leq l \rangle$ and $\langle (\beta_j, f_j) \mid j < l \rangle$ are witnesses to the values of $c^{p_\varepsilon}(\eta, \nu)$ and $c^{p_\varepsilon}(\nu, \tau)$ respectively. Thus $c^p(\eta, \nu) = c^{p_\varepsilon}(\eta, \nu)$ and $c^p(\nu, \tau) = c^{p_\varepsilon}(\nu, \tau)$ and one has $c^p(\eta, \nu) \leq c^p(\nu, \tau)$ as required. Hence $p \in \mathbb{P}$.

In order to show that $p \leq p_\varepsilon$ for $\varepsilon < \chi$ one needs only to show that $i_p(\nu, \tau) = i_{p_\varepsilon}(\nu, \tau)$ for ν, $\tau \in a_{p_\varepsilon}$.

If $\eta \in i_p(\nu, \tau)$ there is some $\varepsilon' \in [\varepsilon, \chi)$ such that $\eta \in a_{p_{\varepsilon'}}$, and hence $\eta \in i_{p_{\varepsilon'}}(\nu, \tau)$ by the definition of $i_{p_{\varepsilon'}}$ (again). But $p_{\varepsilon'} \leq p_\varepsilon$, so $\eta \in a_{p_\varepsilon}(\nu, \tau)$.

Conversely, suppose towards a contradiction that $\eta \in i_{p_\varepsilon}(\nu, \tau)$ and there is some $\xi \in a_p$ with $\eta <_p \xi \leq_p \nu$, τ. Then there is some $\varepsilon' \in [\varepsilon, \chi)$ such that $\xi \in a_{p_{\varepsilon'}}$. Hence $\eta <_{p_{\varepsilon'}} \xi \leq_{p_{\varepsilon'}} \nu$, τ. But this contradicts the fact that $i_{p_{\varepsilon'}}(\nu, \tau) = i_{p_\varepsilon}(\nu, \tau)$, which is true because $p_{\varepsilon'} \leq p_\varepsilon$.

Lemma 16. For each $\xi < \kappa^+$ the collection $\{ p \in \mathbb{P} \mid \xi \in a_p \}$ is a dense (and open) subset of $\mathbb{P}$.

Proof. Let $\xi < \kappa^+$ and $p \in \mathbb{P}$. If $\xi \in a_p$ there is nothing to prove. Otherwise define q by $a_q = a_p \cup \{ \xi \}$, $\leq_q = \leq_p$, $A_q = A_p$. Then $q \in \mathbb{P}$ and $\xi \in a_q$.

Definition 17. Let G be $\mathbb{P}$-generic over V and define $B_\alpha = \{ \beta \leq \alpha \mid \exists p \in G \ \beta \leq_p \alpha \}$ for each $\alpha < \kappa^+$. Let $i(\alpha, \beta) = \{ \gamma \mid \exists p \in G \ \gamma \in i_p(\alpha, \beta) \}$ for $\alpha < \beta < \kappa^+$. Note that $\overline{\overline{i(\alpha, \beta)}} < \mu$ for all $\alpha < \beta < \kappa^+$.

Let $\mathcal{B} = \{ B_\alpha \setminus \bigcup \{ B_\beta \mid \beta \in b \} \mid \alpha < \kappa^+ \ \ \& \ \ b \in [\alpha]^{<\mu} \}$. Let τ be the topology on κ^+ generated by taking $\mathcal{B}$ as a sub-basis.

Lemma 18. $\mathcal{B}$ is a basis for a topology on κ^+, not just a sub-basis.

Proof. Let $\mathcal{O}_0 = B_\alpha \setminus \bigcup \{ B_\gamma \mid \gamma \in a \}$ and $\mathcal{O}_1 = B_\beta \setminus \bigcup \{ B_\varepsilon \mid \varepsilon \in b \}$ be sets in $\mathcal{B}$. Then $\mathcal{O}_0 \cap \mathcal{O}_1 = B_\alpha \cap B_\beta \cap (\kappa^+ \setminus \bigcup \{ B\gamma \mid \gamma \in a \cup b \})$.

Now $B_\alpha \cap B_\beta = \bigcup \{ B_\varepsilon \mid \varepsilon \in i(\alpha, \beta) \}$, so $\mathcal{O}_0 \cap \mathcal{O}_1 = \bigcup \{ B_\varepsilon \setminus \bigcup \{ B\gamma \mid \gamma \in a \cup b \} \mid \varepsilon \in i(\alpha, \beta) \}$. So in order to show that $\mathcal{O}_0 \cap \mathcal{O}_1$ is a union of sets in $\mathcal{B}$ it suffices to show that each $B_\varepsilon \setminus \bigcup \{ B_\gamma \mid \gamma \in a \cup b \}$ is in $\mathcal{B}$.

But $B_\varepsilon \setminus \bigcup \{ B_\gamma \mid \gamma \in a \cup b \} = B_\varepsilon \setminus \bigcup \{ B_\varepsilon \cap B_\gamma \mid \gamma \in a \cup b \}$ and each $B_\varepsilon \cap B_\gamma = \bigcup \{ B_\nu \mid \nu \in i\{\gamma, \varepsilon\} \}$ (recalling that $i\{\gamma, \varepsilon\}$ denotes $i(\gamma, \varepsilon$ if $\gamma \leq \varepsilon$, and $i(\varepsilon, \gamma)$ if instead $\varepsilon \leq \gamma$). Since each $i(\gamma, \varepsilon)$ has size less than μ, one has that $\{ \nu \mid \exists \gamma \in a \cup b \ \nu \in i\{\gamma, \varepsilon\} \}$ is a union of fewer than μ sets each of size less than μ and so, as μ is regular, itself has size less than μ. Thus $B_\varepsilon \setminus \bigcup \{ B_\gamma \mid \gamma \in a \cup b \} \in \mathcal{B}$ as required and so $\mathcal{O}_1 \cap \mathcal{O}_2$ is a union of sets in $\mathcal{B}$.

Lemma 19. $\mathcal{B}$ is a clopen basis and so τ is 0-dimensional.

Proof. If $\mathcal{O} = B_\alpha \setminus \bigcup \{ B_\gamma \mid \gamma \in a \}$ then $\kappa^+ \setminus \mathcal{O} = (\kappa^+ \setminus B_\alpha) \cup \{ B_\gamma \mid \gamma \in a \}$. Clearly $B_\gamma \in \mathcal{B}$ for $\gamma \in a$. Also $\kappa^+ \setminus B_\alpha = \bigcup \{ B_\beta \setminus B_\alpha \mid \beta < \kappa^+ \}$. But $B_\beta \setminus B_\alpha = B_\beta \setminus (B_\beta \cap B_\alpha) = B_\beta \setminus \bigcup \{ B_\nu \mid \nu \in i\{\alpha, \beta\} \}$, as in the proof of Lemma (18). Thus each $B_\beta \setminus B_\alpha$ is a union of sets in $\mathcal{B}$, and hence $\kappa^+ \setminus \mathcal{O}$ is also a union of sets in $\mathcal{B}$.

Lemma 20. τ is right-separated.

Proof. If $\alpha < \kappa^+$ then $\alpha \in B_\alpha$, but $B_\alpha \subseteq \alpha + 1$, so $\beta \notin B_\alpha$ for $\alpha < \beta < \kappa^+$. Of course, each $B_\alpha \in \mathcal{B}$.

Lemma 21. (κ^+, τ) is locally μ-Lindelöf.

Proof. Let $\alpha < \kappa^+$. $\alpha \in B_\alpha$, and B_α is closed (by Lemma (19)). So it suffices to show that each B_α is μ-Lindelöf. This is done by induction on κ^+.

Suppose that B_γ is μ-Lindelöf for each $\gamma < \alpha$. Let $\langle \mathcal{O}_i \mid i \in I \rangle$ be an open cover of B_α. So there is some $i \in I$ and some $B_\beta \setminus \bigcup\{ B_{\beta_j} \mid \beta_j \in b \} \subseteq \mathcal{O}_i$ such that $\alpha \in B_\beta \setminus \bigcup\{ B_{\beta_j} \mid \beta_j \in b \} \in \mathcal{B}$. So B_α has a subcover of size less than μ if and only if $B_\alpha \cap B_{\beta_j}$ has a subcover of size less than μ for each $j \in b$ (using the facts that $\overline{\overline{b}} < \mu$ and that μ is regular). But $B_\alpha \cap B_{\beta_j} = \bigcup\{ B_\gamma \mid \gamma \in i\{\alpha, \beta_j\} \}$ for each $j \in b$, and thus is the union of fewer than μ many sets of the form B_ε with $\varepsilon < \alpha$ since each $i\{\alpha, \beta_j\}$ has size less than μ.

By the inductive hypothesis each $B_\varepsilon, \subseteq B_\alpha \subseteq \bigcup\{ \mathcal{O}_i \mid i \in I \}$, has a subcover of size less than μ, and hence $B_\alpha \cap B_{\beta_j}$ has the union of these subcovers as a subcover of size less than μ. Hence B_α is μ-Lindelöf in τ.

Lemma 22. Suppose $\mathcal{O} \in \mathcal{B}$. Then $\mathcal{O} \cap \mu \neq \emptyset$.

Proof. Let $p \in \mathbb{P}$ be such that $p \Vdash$ "$\mathcal{O} = B_\alpha \setminus \bigcup\{ B_\gamma \mid \gamma \in a \} \in \mathcal{B}$" and suppose, without loss of generality, using the μ-closure of $\mathbb{P}$, that $\{\alpha\} \cup a \subseteq a_p$. Let $\beta \in \mu \setminus a_p$. Define q by $a_q = a_p \cup \{\beta\}$, $A_q = A_p$ and $\leq_q \restriction a_p \times a_p = \leq_p$ and for each $\varepsilon \in a_p$ set $\beta \leq_q \varepsilon$ if and only if $\alpha \leq_p \varepsilon$. It is clear that $q \in \mathbb{P}$ and $q \leq p$. And it is also clear that $q \Vdash \beta \in \mathcal{O} \cap \mu$.

Corollary 23. The separability degree of τ is μ.

Proof. Lemma (22) gives μ as an upper bound, but it is clear from the definition of $\mathbb{P}$ that given any smaller sized set and some condition there is a stronger condition that forces some set in $\mathcal{B}$ to miss it.

Proposition 24. τ has tightness μ.

Proof. Write $cl(Z)$ for the closure of Z in the topology generated by $\mathcal{B}$. I prove by induction that if $\gamma \in \mathrm{cl}(Z)$ for some $Z \subseteq \kappa^+$ then there is some $Y \in [Z]^\mu$ such that $\gamma \in \mathrm{cl}(Y)$. (*cf.* [M*2, Theorem 4.25]).

So fix $Z \subseteq \kappa^+$, let $\gamma \in \mathrm{cl}(Z)$ and assume that for all $\varepsilon < \gamma$ there is First of all $Z \setminus B_\gamma \subseteq \kappa^+ \setminus B_\gamma$, so $\mathrm{cl}(Z \setminus B_\gamma) \subseteq \mathrm{cl}(\kappa^+ \setminus B_\gamma) = \kappa^+ \setminus B_\gamma$, since B_γ is open in τ. Thus $\mathrm{cl}(Z \setminus B_\gamma) \cap B_\gamma = \emptyset$.

This gives us that $\gamma \in \mathrm{cl}(Z \cap B_\gamma)$, because $\mathrm{cl}(Z) = \mathrm{cl}(Z \cap B_\gamma) \cup \mathrm{cl}(Z \setminus B_\gamma)$ since $\mathrm{cl}(.)$ is a closure operator.

Now B_γ is closed in τ by Lemma (19), so $\mathrm{cl}(Z \cap B_\gamma) \subseteq B_\gamma$. I now proceed by a series of claims.

Claim 25. There is some $Y \subseteq Z$ such that $\gamma \in \mathrm{cl}(Y)$ and $\mathrm{cl}(Y) \cap \gamma$ has no maximal element.

Proof. Let $X_0 = Z \cap B_\gamma$ and inductively set δ_i to be maximal in $\mathrm{cl}(X_i) \cap \gamma$ and $X_{i+1} = X_i \setminus B_{\delta_i}$ for as long as possible. By the above observations applied to X_i in place of Z, $\mathrm{cl}(X_{i+1}) \subseteq (\kappa^+ \setminus B_{\delta_i})$ and $\mathrm{cl}(X_i \cap B_{\delta_i}) \subseteq B_{\delta_i}$, so that $\gamma \in \mathrm{cl}(X_{i+1}) \setminus \mathrm{cl}(X_i \cap B_{\delta_i})$ and $\delta_i \in \mathrm{cl}(X_i \cap B_{\delta_i}) \setminus \mathrm{cl}(X_{i+1})$ for each i for which δ_i is defined. As $X_{i+1} \subseteq X_i$ one has that $\delta_{i+1} \leq \delta_i$ for all i for which δ_{i+1} is defined. But $\delta_{i+1} \in \kappa^+ \setminus B_{\delta_i}$, so $\delta_{i+1} \neq \delta_i$ and, hence, $\delta_{i+1} < \delta_i$ for each i for which δ_{i+1} is defined. So let $k < \omega$ be least such that $\mathrm{cl}(X_k) \cap \gamma$ has no maximal element. Set $Y = X_k$. Then $Y \subseteq X_0 = Z$, $\gamma \in \mathrm{cl}(Y)$ and $\mathrm{cl}(Y) \cap \gamma$ has no maximal element.

Claim 26. There is some $Y' \in [\mathrm{cl}(Y) \cap \gamma]^\mu$ such that $\gamma \in \mathrm{cl}(Y')$.

Proof. Write W for $\mathrm{cl}(Y) \cap \gamma$, and set $W^* = \mu \cap \bigcup\{B_\delta \mid \delta \in W\}$. Note that $Y \subseteq W$ since $\mathrm{cl}(Y) \subseteq B_\gamma$ and $\gamma \notin Y$, and that $\mathrm{cl}(W) = \mathrm{cl}(Y)$. Choose $Y' \in [W]^\mu$ so that $W^* \subseteq \bigcup\{B_\delta \mid \delta \in Y'\}$.

Subclaim 27. $\gamma \in \mathrm{cl}(W^*)$.

Proof. Suppose that $\mathcal{O}$ is an open set and that $\mathcal{O} \cap W^* = \emptyset$. Then for all $\delta \in W$ one has that $\mathcal{O} \cap B_\delta \cap \mu = \emptyset$. By Lemma (22) this implies that $\mathcal{O} \cap B_\delta = \emptyset$ for all $\delta \in W$. Hence $\mathcal{O} \cap \bigcup\{B_\delta \mid \delta \in W\} = \emptyset$. But $t \in \mathrm{cl}(Y) = \mathrm{cl}(W) \subseteq \mathrm{cl}(\bigcup\{B_\delta \mid \delta \in W\}$, so one must have $t \notin \mathcal{O}$.

Since $\mathrm{cl}(Y') \subseteq \mathrm{cl}(Y) \subseteq B_\gamma$, one has that $\mathrm{cl}(Y')$ is μ-Lindelöf in τ. Suppose that $\gamma \notin \mathrm{cl}(Y')$.

Then $\{B_\delta \mid \delta \in W\}$ is a cover of $\mathrm{cl}(Y')$ and so there is some $\chi < \mu$ and some $\{\varepsilon_i \mid i < \chi\}$ such that $\mathrm{cl}(Y') \subseteq \bigcup\{B_{\varepsilon_i} \mid i < \chi\}$. Hence if $\delta \in \mathrm{cl}(Y')$ one has that there is some $i < \chi$ such that $\delta \in B_{\delta_i}$, and so $B_\delta \subseteq B_{\varepsilon_i}$.

Thus $\bigcup\{B_\delta \mid \delta \in W\} \subseteq \bigcup\{B_{\varepsilon_i} \mid i < \chi\}$. Consequently $W^* \subseteq \bigcup\{B_{\varepsilon_i} \mid i < \chi\}$ and so, trivially, $\mathrm{cl}(W^*) \subseteq \mathrm{cl}(\bigcup\{B_{\varepsilon_i} \mid i < \chi\})$.

However, $B_\gamma \setminus \bigcup\{B_{\varepsilon_i} \mid i < \chi\}$ is an open set in τ which contains γ and has empty intersection with $\bigcup\{B_{\varepsilon_i} \mid i < \chi\}$, and thus shows that $\gamma \notin \mathrm{cl}(\bigcup\{B_{\varepsilon_i} \mid i < \chi\})$, and hence that $\gamma \notin \mathrm{cl}(W^*)$, contradicting Subclaim (27). Thus $\gamma \in \mathrm{cl}(Y')$ as required.

Conclusion of of proof of Proposition (24). If $\delta \in Y'$ then by the inductive hypothesis there is some $W_\delta \in [Z]^\mu$ such that $\delta \in \mathrm{cl}(W_\delta)$. Set $Y'' = \bigcup\{W_\delta \mid \delta \in Y'\}$. Then $\overline{\overline{Y''}} \leq \mu$ and $Y' \subseteq \mathrm{cl}(Y'')$, and so $\gamma \in \mathrm{cl}(Y') \subseteq \mathrm{cl}(\mathrm{cl}(Y'')) = \mathrm{cl}(Y'')$. then

By Lemmas (18) to (23) and Proposition (24), (κ^+, τ) has the topological properties described in Theorem (2) and cardinals are preserved in the generic extension by Propositions (5), (13) and (15).

Comment. As remarked after the statement of Theorem (2) it seems unlikely that one can force to add a 0-dimensional, locally compact, countably tight, scattered space of size κ^+ by κ-$\mathbb{M}$-proper forcing when μ is uncountable. It is hard to see how to amalgamate two conditions whose working parts are infinite, albeit of size less than μ, and keep the size of the ranges of an analogue of i_r finite. Of course the consistency of there being such spaces is a major, and radically open, problem.

Chains in $\mathcal{P}(\kappa)$ mod $<\mu$.

Theorem 28. If there is a stationary $(\kappa, 1)$-simplified morass with and κ is the successor of a regular cardinal μ with $\kappa^{<\mu} = \kappa$ then there is a cardinal preserving forcing which adds a chain of length κ^+ in $\mathcal{P}(\kappa)$ mod $< \mu$.

In this subsection I show how to use κ-$\mathbb{M}$-proper forcing to add a chain of length κ^+ in $\mathcal{P}(\kappa)$ mod $<\mu$, that is a sequence $\langle X_\nu \mid \nu < \kappa^+ \rangle$ of subsets of κ such that $\overline{\overline{X_\nu \setminus X_\tau}} < \mu$ and $\overline{\overline{X_\tau \setminus X_\nu}} = \kappa$ for each $\nu < \tau < \kappa^+$. One has that if the Chang conjecture $(\kappa^{++}, \kappa^+) \longrightarrow (\kappa^+, \kappa)$ holds then there is no such chain — Koszmider's proof ([K98, Fact 1]) that this is so when $\kappa = \omega_1$ goes through for arbitrary successor κ. The forcing is the obvious generalisation of Koszmider's forcing from [K98] with sets of maps from the simplified morass of size less than μ as side conditions.

Definition 29. Let the forcing $\mathbb{P}$ consist of conditions $p = (a_p, b_p, f_p, A_p)$ where $a_p \in [\kappa^+]^{<\mu}$, $b_p \in [\kappa]^{<\mu}$, $f_p : a_p \times b_p \longrightarrow 2$ and $A_p \in [\mathcal{F}]^{<\mu}$ and, writing c^p for c_{A_p},

$$(\bullet) \qquad \forall \nu, \tau \in a_p \; \forall \beta \in b_p \; \big((\nu < \tau \;\;\&\;\; c^p(\nu, \tau) \leq \beta) \longrightarrow (f_p(\nu, \beta) = 1 \longrightarrow f_p(\tau, \beta) = 1)\big).$$

$q \leq p$ if $a_p \subseteq a_q$, $b_p \subseteq b_q$, $f_p = f_q \restriction a_p \times b_p$, $A_p \subseteq A_q$ and

$$(\circ) \qquad \forall \nu, \tau \in a_p \; \forall \beta \in b_q \setminus b_p \; \big(\nu < \tau \longrightarrow (f_q(\nu, \beta) = 1 \longrightarrow f_q(\tau, \beta) = 1)\big).$$

The idea behind the forcing is that if G is $\mathbb{P}$-generic over the ground model then one gets the desired chain in $\mathcal{P}(\kappa)$ mod $<\mu$ by setting $X_\nu = \{\beta < \kappa \mid \exists p \in G \; f_p(\nu, \beta) = 1\}$ for $\nu < \kappa^+$. The following two lemmas say that this will happen.

Lemma 30. $\{p \in \mathbb{P} \mid \beta \in b_p\}$ is dense and open for each $\beta < \kappa$.

Proof. If $\beta \notin b_p$ then define q by setting $a_q = a_p$, $A_q = A_p$, $b_q = b_p \cup \{\beta\}$, $f_q \restriction a_p \times b_p = f_p$ and $f_q(\nu, \beta) = 0$ for all $\nu \in a_p$. Then clearly $q \in \mathbb{P}$, $q \leq p$ and $q \in \{p \in \mathbb{P} \mid \beta \in b_p\}$.

Lemma 31. $\{p \in \mathbb{P} \mid \tau \in a_p\}$ is dense and open for each $\tau < \kappa^+$.

Proof. If $\tau \notin a_p$ then define q by setting $a_q = a_p \cup \{\tau\}$, $b_q = b_p$, $A_q = A_p$, $f_q \restriction a_p \times b_p = f_p$, and by setting, for each $\beta \in b_p$, $f_q(\tau, \beta) = 1$ if there is some $\nu \in a_p \cap \tau$ such that $c^q(\nu, \tau) \leq \beta$ and $f_p(\nu, \beta) = 1$, and $f_q(\tau, \beta) = 0$ otherwise. If $\beta \in b_p$, and $f_q(\tau, \beta) = 1$ then by the definition just given of f_q there is some $\nu < \tau$ such $c^q(\nu, \tau) \leq \beta$ and $f_p(\nu, \beta) = 1$. If one also has $\xi \in a_p \setminus \tau$ and $c^q(\tau, \xi) \leq \beta$ then $c^q(\nu, \xi) \leq \beta$ by the subadditivity of c^q. As $c^q = c^p$, the condition $(\bullet)$ for p applied to ν and ξ ensures that $f_p(\xi, \beta) = 1$. This shows that the condition $(\bullet)$ also holds for q, so that $q \in \mathbb{P}$. It is clear that $q \leq p$ (the condition $(\circ)$ holds vacuously since $b_q \setminus b_p = \emptyset$).

It remains to show that $\mathbb{P}$ preserves cardinals.

Lemma 32. $\mathbb{P}$ is μ-closed.

Proof. Let $\langle p_i \mid i < \chi \rangle$ be a descending sequence of conditions for some limit ordinal $\chi < \mu$. Define p by setting $a_p = \bigcup\{a_{p_i} \mid i < \chi\}$, $b_p = \bigcup\{b_{p_i} \mid i < \chi\}$, $f_p = \bigcup\{f_{p_i} \mid i < \chi\}$, and $A_p = \bigcup\{A_{p_i} \mid i < \chi\}$.

First of all I show that ($\bullet$) holds for p and hence that $p \in \mathbb{P}$. Suppose that ν, $\tau \in a_p$, $\beta \in b_p$, $c^p(\nu, \tau) \leq \beta$ and $f_p(\nu, \beta) = 1$. Let $c^p(\nu, \tau) = \alpha$ and let $\langle \xi_j \mid j \leq k \rangle$ and $\langle (\gamma_j, g_j) \mid j < k \rangle$ witness this. Then there is some $i < \chi$ such that ν, $\tau \in a_{p_i}$, $\beta \in b_{p_i}$ and $(\gamma_j, g_j) \in A_{p_i}$ for each $j < k$. So $f_{p_i}(\nu, \beta) = 1$ and $\langle \xi_j \mid j \leq k \rangle$ and $\langle (\gamma_j, g_j) \mid j < k \rangle$ witness that $c^{p_i}(\nu, \tau) \leq \alpha$. (As $A_{p_i} \subseteq A_p$ this in fact gives that $c^{p_i}(\nu, \tau) = c^p(\nu, \tau)$ by Lemma (1.3).) By ($\bullet$) for p_i one has that $f_{p_i}(\tau, \beta) = 1$, and hence $f_p(\tau, \beta) = 1$.

Next I show that $p \leq p_i$ for all $i < \chi$ by checking that ($\circ$) holds between p and each p_i. So let $i < \chi$ with ν, $\tau \in a_{p_i}$, $\nu < \tau$, $\beta \in b_p \setminus b_{p_i}$ and $f_p(\nu, \beta) = 1$. Then there is some $i' \in [i, \chi)$ such that $\beta \in p_{i'}$, whence $f_{p_{i'}}(\nu, \beta) = 1$ since $f_p \restriction a_{p_{i'}} \times b_{p_{i'}} = f_{p_{i'}}$. By ($\circ$) for $p_{i'}$ and p_i one has that $f_{p_{i'}}(\tau, \beta) = 1$ and hence that $f_p(\tau, \beta) = 1$, as required.

Lemma 33. $\mathbb{P}$ has the κ^+-chain condition.

Proof. Let $\{p_i \mid i < \kappa^+\}$ be a subset of $\mathbb{P}$ of size κ^+. For each $p' \in \{p'_i \mid i < \kappa^+\}$, by Fact (I.5), let $(\gamma, g) \in \mathcal{F}$ be such that $a_{p'} \subseteq \mathrm{rge}(g)$, $\mathrm{ssup}(b_{p'}) < \gamma$ and for all $(\beta, f) \in A_{p'}$ there is some $f' \in \mathcal{F}_{\beta\gamma}$ with $f = g \cdot f'$. Define a condition p by letting $a_p = a_{p'}$, $b_p = b_{p'}$, $f_p = f_{p'}$ and $A_p = A_{p'} \cup \{(\gamma, g)\}$. Now it is clear that $c^{p'}(\nu, \tau) \leq \beta$ if and only if $c^p(\nu, \tau) \leq \beta$ for any pair ν, $\tau \in a_p$ and $\beta \in b_p$ since any sequence of maps witnessing the latter cannot contain (γ, g) (as $\beta < \gamma$) and, hence, witnesses the former. Hence p satisfies ($\bullet$) and so is a condition. As ($\circ$) holds vacuously and all of the other conditions for being an extension hold by definition one also has that $p \leq p'$. If one can find i, $j < \kappa^+$ such that p_i and p_j are compatible then, clearly, one will have that p'_i and p'_j are compatible as well.

By thinning using the Δ-system lemma if necessary one may as well assume that there is some $b \in [\kappa]^{<\mu}$ such that $b = b_{p_i}$ for all $i < \kappa^+$ since $\kappa^{<\mu} = \kappa$. One may as well also assume there is some $a \in [\kappa^+]^{<\mu}$ such that a is an initial segment of each a_{p_i}, that if $i < j < \kappa^+$ then $\max(a_{p_i}) < \min(a_{p_j} \setminus a)$, and there is an order preserving isomorphism from a_{p_i} to a_{p_j} which extends to an isomorphism from f_{p_i} to f_{p_j}. Moreover, one may as well assume that there is some $\alpha < \kappa$, some $A \in [\bigcup\{\{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha\}]^{<\mu}$ and a map $F_i \in \mathcal{F}_{\alpha\kappa}$ for each $\nu < \kappa^+$ such that $A_{p_i} = \{(\beta, F_\nu \cdot h) \mid (\beta, h) \in A\} \cup \{(\alpha, F_i)\}$ and such that $F_i \cdot F_j^{-1} \restriction a_{p_j}$ is the order preserving isomorphism from (a_{p_j}, f_{p_j}) to (a_{p_i}, f_{p_i}). This is possible using the Δ-system lemma for morass maps, Proposition (1.13). Clearly for each $i < \kappa^+$ and $\xi < \zeta < \kappa^+$ one has that $c^{p_i}(\xi, \zeta) \leq \alpha$ if and only if $c^{p_i}(\xi, \zeta) < \kappa$.

Let $i < j < \kappa^+$ and set $a_r = a_{p_i} \cup a_{p_j}$, $b_r = b$, $f_r \restriction a_{p_i} \times b = f_{p_i}$ and $f_r \restriction a_{p_j} \times b = f_{p_j}$, and $A_r = A_{p_i} \cup A_{p_j}$.

It is also clear that if $\xi < \zeta < \kappa^+$ then $c^r(\xi, \zeta) \leq \alpha$ if and only if $c^r(\xi, \zeta) < \kappa$. By Lemma (1.14), one has that if $\zeta \in \mathrm{rge}(F_i)$ then $c^r(\xi, \zeta) = c^{p_i}(\xi, \zeta)$ and if $\zeta \in \mathrm{rge}(F_j)$ then $c^r(\xi, \zeta) = c^{p_j}(\xi, \zeta)$.

Claim 34. If $\xi \in a_{p_i} \setminus a_{p_j}$ then $\xi \notin \operatorname{rge}(F_j)$.

Proof. If $\xi \in \operatorname{rge}(F_j)$ then $\xi \in \operatorname{rge}(F_j) \cap \operatorname{rge}(F_i)$ since $F_j \cdot F_i^{-1} \restriction a_{p_i}$ is the order preserving isomorphism from a_{p_i} to a_{p_j}. But then one has that $F_j(\xi_\alpha) = \xi = F_i(\xi_\alpha)$ by Fact (1.3), so $F_j \cdot F_i^{-1}(\xi) = \xi$ and hence $\xi \in a_{p_i} \cap a_{p_j}$, a contradiction.

It is now immediate that if $\xi \in a_{p_i} \setminus a_{p_j}$ and $\zeta \in a_{p_j} \setminus a_{p_i}$ then $\xi \notin \operatorname{rge}(F_j)$, by Claim (34), so that $c^{p_i}(\xi, \zeta) = \kappa$, and thus $c^r(\xi, \zeta) = \kappa$ from Lemma (1.14). Hence for such ξ, ζ one has that $\beta < c^r(\xi, \zeta)$ for all $\beta \in b$. Hence $(\bullet)$ is vacuously true for all such pairs ξ, ζ.

By Claim (34) as well, if ξ_0, $\xi_1 \in a_{p_i}$ then $c^r(\xi_0, \xi_1) = c^{p_i}(\xi_0, \xi_1)$. And so if $\xi_0 < \xi_1$, $c(\xi_0, \xi_1) \leq \beta$ and $f_r(\xi_0, \beta) = 1$ for some $\beta \in b$ then (by $(\bullet)$ for p_i) one also has that $f_r(\xi_1, \beta) = 1$. Similarly, if ζ_0, $\zeta_1 \in a_{p_j}$, $c(\xi_0, \xi_1) \leq \beta$ and $f_r(\zeta_0, \beta) = 1$ for some $\beta \in b$ then (by $(\bullet)$ for p_j) one also has that $f_r(\zeta_1, \beta) = 1$.

Hence $r \in \mathbb{P}$, and thus $r \leq_{\mathbb{P}} p_i$, p_j (since $(\circ)$ holds vacuously). So $\mathbb{P}$ does have the κ^+-cc.

The last two lemmas show that cardinals below κ and above κ, respectively, are preserved. So in order to complete the proof it suffices to show that $\mathbb{P}$ is κ-$\mathbb{M}$-proper and hence that κ itself is preserved.

Proposition 35. $\mathbb{P}$ is κ-$\mathbb{M}$-proper.

Proof. The proof strategy will be to work through the framework of §4 checking that one can actually perform each of the four steps enumerated there for $\mathbb{P}$.

So suppose that $p \in \mathbb{P}$ and that $\mathcal{N}$ is as in the general framework but for this specific $\mathbb{P}$. Obviously, a_p is the realm of p and A_p is the side condition part.

Step A. Checking that $(a_p, b_p, f_p, A_p \cup \{(\delta, F)\})$ is a condition and is stronger than p.

Proof. This is just as in the argument of the first paragraph of the proof of the κ^+-chain condition (Lemma (33)). Let $p' = (a_p, b_p, f_p, A_p \cup \{(\delta, F)\})$. Clearly p' is a condition if it satisfies $(\bullet)$ and is stronger than p if it is a condition because $(\circ)$ will hold vacuously. But if ν, $\tau \in a_p$, $\beta \in b_p$ and $c^{p'}(\nu, \tau) \leq \beta$ then any maps in a witnessing sequence to this cannot include (δ, F) because $p \in \mathcal{N}$ and has size less than μ, so $b_p \subseteq \mathcal{N} \cap \kappa = \delta$. Hence any maps from such a witnessing sequence witness that $c^p(\nu, \tau) \leq \beta$ as well. Thus if $p(\nu, \beta) = 1$ one also has that $p(\tau, \beta) = 1$ by $(\bullet)$ for p.

Now let $\mathcal{D}$ be a dense, open subset of $\mathbb{P}$ and $q \in \mathcal{D}$ with $q \leq p'$.

Step B. One may assume without loss of generality that $c^q(\nu, \tau) < \kappa$ for all ν, $\tau \in a_q$

Proof. The argument is almost exactly the same argument as in the first paragraph of the proof of the κ^+-chain condition (Lemma (33)) and the argument used for Step (A). Explicitly, suppose one has a q without this property. Then for each $\nu \in a_q$ choose a map $(\gamma_\nu, g_\nu) \in \mathcal{F}$ such that $\nu \in \operatorname{rge}(g_\nu)$. Now apply Fact (I.5) to this collection of maps and select some $(\gamma, g) \in \mathcal{F}$ through which they all factor and with γ greater than $\mathrm{ssup}(\{\alpha \mid \exists(\alpha, f) \in A_q\} \cup b_p)$. Add this single map (γ, g) to A^q. As in the argument for Step (A) the resulting object is still a condition since the new map cannot play a rôle in witnessing that $c(\nu, \tau) \leq \beta$ for any $\beta \in b_q$ and ν, $\tau \in a_q$.

Next let $q \restriction \mathcal{N} = (a_q \cap \mathcal{N}, b_q \cap \mathcal{N}, f_q \cap \mathcal{N}, A_q \cap \mathcal{N})$. Note that $f_q \cap \mathcal{N} = f_q \restriction (a_q \cap \mathcal{N} \times b_q \cap \mathcal{N})$, since the range of f_q is (a subset of) $\{0, 1\}$. This makes the proof that $q \restriction \mathcal{N}$ particularly simple.

Step C. Show that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and that $q \leq q \restriction \mathcal{N}$.

Proof. Each of the four components is a set of size less than μ which is an element of $\mathcal{N}$ and so is itself an element of $\mathcal{N}$ since the latter is closed under sequences of length less than μ. Thus $q \restriction \mathcal{N} \in \mathcal{N}$.

Now I prove that $q \restriction \mathcal{N} \in \mathbb{P}$ by showing that $(\bullet)$ holds. Let ν, $\tau \in a_q \cap \mathcal{N}$, $\beta \in b_q \cap \mathcal{N} (= b_q \cap \delta)$, $\nu < \tau$, $c^{q \restriction \mathcal{N}}(\nu, \tau) \leq \beta$ and $f_q(\nu, \beta) = 1$. By Lemma (1.3) one has that $c^q(\nu, \tau) \leq c^{q \restriction \mathcal{N}}(\nu, \tau) \leq \beta$, so one has that $f_q(\tau, \beta) = 1$ by $(\bullet)$ for q. Thus $q \restriction \mathcal{N}$ does satisfy $(\bullet)$ and so $q \restriction \mathcal{N} \in \mathbb{P}$.

Finally I prove that $q \leq q \restriction \mathcal{N}$ by demonstrating that $(\circ)$ holds. Let ν, $\tau \in a_q \cap \mathcal{N}$, again, with $\nu < \tau$, let $\beta \in b_q \setminus \mathcal{N}$ and suppose that $f_q(\nu, \beta) = 1$. As ν, $\tau \in \mathcal{N} = \mathrm{rge}(F)$, $(\delta, F) \in A_q$ and $\beta \notin \mathcal{N}$ one has that $c^q(\nu, \tau) \leq \delta \leq \beta$. Hence, by $(\bullet)$ for q, one has that $f_q(\tau, \beta) = 1$. Thus $(\circ)$ does hold between q and $q \restriction \mathcal{N}$ and one does have that $q \leq q \restriction \mathcal{N}$.

Now obtain s as in §4 using b_q as the arbitrary set e_q in the section of Notation in §4.

Step D. Define some r with realm $a_s \cup a_q$ and side condition part $A_s \cup A_q$. Show that $r \in \mathbb{P}$ and that $r \leq s$, q.

Proof. Write a for $a_s \cap a_q$, a_0 for $a_s \setminus a_q$, a_1 for $a_q \setminus a_s$, b for $b_s \cap b_q$, b_0 for $b_s \setminus b_q$, b_1 for $b_q \setminus b_s$.

Define r as follows. Let $a_r = a_s \cup a_q$, $b_r = b_s \cup b_q$ and $A_r = A_s \cup A_q$. If $(\tau, \beta) \in a_s \times b_s$ let $f_r(\tau, \beta) = f_s(\tau, \beta)$ and if $(\tau, \beta) \in a_q \times b_q$ let $f_r(\tau, \beta) = f_q(\tau, \beta)$. It remains to define $f_r \restriction a_0 \times b_1 \cup a_1 \times b_0$. This is done separately for each β and by induction on τ.

If $\beta \in b_1$ and $\tau \in a_0$ let $f_r(\tau, \beta) = 1$ if there is some $\nu \in a_r \cap \tau$ such that $c^r(\nu, \tau) \leq \beta$ and $f_r(\nu, \beta) = 1$ or if there is some $\nu \in a_0 \cap \tau$ such that $f_r(\nu, \beta) = 1$. Otherwise set $f_r(\nu, \tau) = 0$.

Similarly, if $\beta \in b_0$ and $\tau \in a_1$ let $f_r(\tau, \beta) = 1$ if there is some $\nu \in a_r \cap \tau$ such that $c^r(\nu, \tau) \leq \beta$ and $f_r(\nu, \beta) = 1$ or there is some $\nu \in a_1 \cap \tau$ such that $f_r(\nu, \beta) = 1$. Otherwise set $f_r(\nu, \tau) = 0$.

Observe that if $\tau \in a_1$ and $f_r(\tau, \beta) = 1$ because there is some $\nu \in a_1$ such that the inductive process already set $f_r(\nu, \beta) = 1$ then there is some such ν (obtainable by taking the least such ν) for which there is some $\zeta \in a_s$ with $c^r(\zeta, \nu) < \beta$ and $f_s(\zeta, \beta) = 1$. A similar observation is true for a_0. These observations are used repeatedly below, without explicit comment.

I show, first of all, that r is a condition by proving that it satisfies $(\bullet)$. Suppose that ν, $\tau \in a_r$ with $\nu < \tau$, $\beta \in b_r$, $c^r(\nu, \tau) \leq \beta$ and $f_r(\nu, \beta) = 1$. If $\tau \in a_1$ and $\beta \in b_0$ or $\tau \in a_0$ and $\beta \in b_1$ then $f_r(\tau, \beta) = 1$ by the definition of f_r just made.

Case A. $\tau \in a_q$ and $\beta \in b_q$. Then there is some $\xi \in a_q \cap \tau$ such that $f_r(\xi, \beta) = 1$. For if $\nu \in a_q$ then take $\xi = \nu$. Otherwise, if $\nu \in a_0$ then there is some $\zeta \leq \nu$ such that $\zeta \in a_0$ and some $\xi \in a_q \cap \zeta$ such that $c^r(\xi, \zeta) \leq \beta$ and $f_r(\xi, \beta) = 1$. As ν, $\zeta \in a_s$ one has that $c^r(\zeta, \nu) < \delta$ (by (4.12)), so $c^r(\xi, \tau) \leq \beta$ by the subadditivity of c^r (Theorem (1.12)).

Subcase A.i. $\beta \in b$. Then $\beta < \alpha^*$ (by clause (iii) of the definition of ϕ for s) and one has that $c^r(\xi, \tau) = c^q(\xi, \tau)$ by (4.15). Thus $f_q(\tau, \beta) = 1$ by ($\bullet$) for q and so $f_r(\tau, \beta) = 1$.

Subcase A.ii. $\beta \in b_1$. One has that $\delta \leq \beta$ (by clause (iii.5) of the definition of ϕ from the Notation of §4). Hence, by (4.16), one has that $c^q(\xi, \tau) \leq \beta$ So one has that $f_q(\tau, \beta) = 1$ by ($\bullet$) for q and hence $f_r(\xi, \tau) = 1$.

Case B. $\tau \in a_s$ and $\beta \in b_s$. One cannot have that $\nu \in a_1$ because then one would have $\beta < \delta \leq c^r(\xi, \tau)$, by (4.17). So suppose that $\nu \in a_s$. Then $c^s(\nu, \tau) = c^r(\nu, \tau) \leq \beta < \delta$ by (4.12), and so $f_s(\tau, \beta) = 1$ by ($\bullet$) for s, whence $f_r(\tau, \beta) = 1$ again.

Now I show that $r \leq q$. Let ν, $\tau \in a_q$, $\nu < \tau$, $\beta \in b_0$ and $f_r(\nu, \beta) = 1$. If $\tau \in a_0$ then $f_r(\tau, \beta) = 1$ by the definition of $f_r(\,.\,, \beta)$. So suppose that $\tau \in a_s \cap a_q$.

Case A. $\nu \in a_s \cap a_q$. Then ν, $\tau \in \mathrm{rge}(h^*)$. This shows that $c^r(\nu, \tau) \leq \beta < \delta$ for otherwise one has $\beta < c^r(\nu, \tau) \leq \alpha^*$, but $b_s \setminus b_q \cap \alpha^* = \emptyset$, a contradiction. Now, $c^s(\nu, \tau) = c^r(\nu, \tau) \leq \beta$, by (4.12), so $f_s(\tau, \beta) = 1$ by ($\bullet$) for s, and hence $f_r(\tau, \beta) = 1$.

Case B. $\nu \in a_1$. Let $\xi \in a_1 \cap \nu$ and $\zeta \in a_s \cap \xi$ be such that $c^r(\zeta, \xi) \leq \beta$ and $f_r(\zeta, \beta) = 1$. If $\zeta \in a_s \cap a_q$ then Case (A) deals with the situation. If $\zeta \in a_0$ then $c^s(\zeta, \tau) = c^r(\zeta, \tau) \leq c^r(\zeta, \xi)$, by (4.12) and (4.22) respectively. So $c^s(\zeta, \tau) \leq \beta$, and $f_s(\tau, \beta) = 1$ by ($\bullet$) for s. Hence $f_r(\tau, \beta) = 1$.

Finally, I show that $r \leq s$. Let ν, $\tau \in a_s$, $\nu < \tau$, $\beta \in b_1$ and $f_r(\nu, \beta) = 1$. If $\tau \in a_1$ then $f_r(\tau, \beta) = 1$ by the definition of $f_r(\,.\,, \beta)$. So suppose that $\tau \in a_s \cap a_q$.

Case A. $\nu \in a_s \cap a_q$. Then $c^q(\nu, \tau) \leq \delta \leq \beta$ since ν, $\tau \in \mathrm{rge}(F)$ so one has that has that $f_q(\tau, \beta) = 1$ by ($\bullet$) for q, whence $f_r(\tau, \beta) = 1$.

Case B. $\nu \in a_0$. Let $\zeta \in a_0 \cap \nu$ and $\xi \in a_s \cap \zeta$ be such that $c^r(\xi, \zeta) \leq \beta$ and $f_r(\xi, 1) = 1$. If $\xi \in a_s \cap a_q$ then one is back in Case (A) again and so finished. If $\xi \in a_1$ then $\delta \leq c^r(\xi, \zeta) = c^r(\xi, \nu) = c^q(\xi, \nu)$ by (4.17), the subadditivity of c^r and the fact that $c^r(\zeta, \nu) < \delta$ by (4.12). But, again by (4.12), $c^r(\nu, \tau) < \delta$. So $\delta \leq c^r(\xi, \nu) = c^r(\xi, \tau) = c^q(\xi, \tau)$, by the subadditivity of c^r once more. Thus $c^q(\xi, \tau) \leq \beta$, and $f_r(\tau, \beta) = 1$ by ($\bullet$) for q. Hence $f_r(\xi, \tau) = 1$.

This concludes the proof that $\mathbb{P}$ is κ-$\mathbb{M}$-proper and so the proof of Theorem (28).

Question. Is it consistent that there are (κ^+, κ^+)-gaps in $\mathcal{P}(\kappa)$ mod $<\mu$? It is believed that this is unknown even in the case $\kappa = \omega_1$.

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