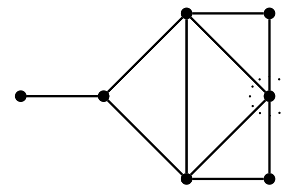
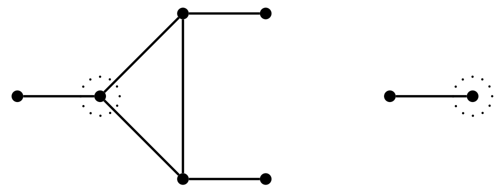
Math 3502: Combinatorial Optimisation. Solutions 6.

1. Applying **MSIS1**, taking the circled vertex as our initial v, to

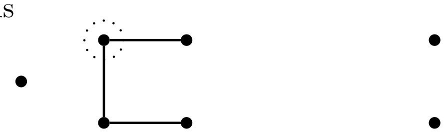


gives us the two graphs

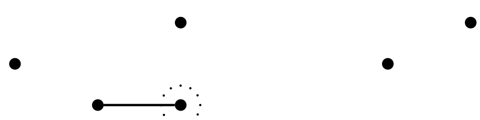


Applying the algorithm to the right hand graph (the single edge) gives the graphs consisting of a single vertex and the empty graph (no vertices or edges), respectively. Neither graph has any edges so, By its base case, **MSIS1** returns 1, 0 respectively for them. Thus on the single edge graph **MSIS1** returns $\max(\{1, 0+1\})$, $= 1$.

Applying the recursive procedure of the alogrithm to the left hand graph above at the circled vertex gives the two graphs

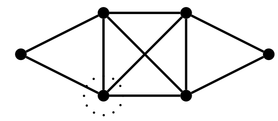


The base case of **MSIS1** applied to the right hand graph (consisting of two isolated vertexes) returns the value 2. On the left hand graph the recursion applied at the circled vertex gives the two graphs

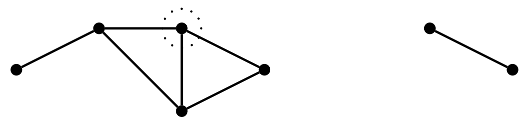


Applying the recursion step one more time to the graph consisting of two isolated points and an edge at the circled vertex gives us the graphs consisting of three and two isolated points respectively. So, by the base case, **MSIS1** returns values of 3, 2, respectively, on these two graphs. By the recursion one thus has that **MSIS1** returns $\max(\{3, 2+1\})$, $= 3$, on the graph consisting of two isolated points and a single edge. On the graph consisting of two isolated points it again returns a value of 2, and so by the recursion returns a value of $\max(\{3, 2+1\})$, $= 3$, on the graph with six vertices. Finally, then, it returns a value of $\max(\{3, 1+1\})$, $= 3$ on our original graph.

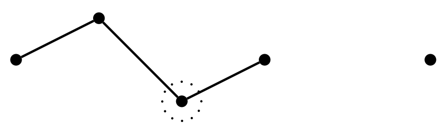
2. Applying **MSIS2**, taking the circled vertex as our initial v, to



gives the two graphs



The base case for **MSIS2** immediately returns the value 1 for the right hand graph (consisting of a single edge. (*cf.* the previous question where **MSIS1** required one more use of the recursion step on such graphs.) For the left hand graph the algorithm produces two further graphs on applying the recursive step:



MSIS2 returns a value of 1 on the isolated vertex and by its recursive step breaks the chain into two further graphs

Now **MSIS2**, by its base case immediately returns $\mid X \mid - \mid E \mid, = 2$, on the graph consisting of an edge and a point, and returns 1 on the isolated vertex. Next, by the recursion, it returns the value $\max(2, 1+1), = 2$, on the chain of length 3. Hence on the graph with 5 vertices it returns $\max(2, 1+1), = 2$. Finally it returns $\max(2, 1+1), = 2$ on the original graph.

3. (a) G is either (empty or) a single isolated vertex or a chain with at least two vertices or a loop (aka circuit) with at least three vertices.

Proof. If $G = (X, E)$ has a vertex of degree 0 it consists solely of this vertex by connectedness. Otherwise all of the vertices of G have degree 1 or 2. If x_0 has degree 1 then there is a unique x_1 such that $\{x_0, x_1\} \in E$. Either $deg(x_1) = 1$ or there is a unique x_2 such that $x_2 \neq x_0$ and $\{x_1, x_2\} \in E$. Continue in this way. If G is finite eventually we find some x_n with $deg(x_n) = 1$. Otherwise we continue forever. Thus G is a chain.

Otherwise every vertex of G has degree 2. Pick x_0, $x_1 \in G$ such that $\{x_0, x_1\} \in E$. Since $deg(x_1) = 2$ there is some unique x_2 such that $\neq x_0$ and $\{x_1, x_2\} \in E$. Continue to choose distinct x_i in a similar way. As G is connected and if G is finite we eventually enumerate all of the vertices - suppose there are $n + 1$ of them. We must then have $\{x_n, x_0\} \in E$, since their degrees are 2. Otherwise we again continue forever alternately defining x_i and x_{-i}, with $\{x_i, x_{i+1}\} \in E$ for all $i \in \mathbb{Z}$.

(b) Such a graph consists of a collection of isolated vertices, chains and circuits. If the graph is finite the independence number of the graph is the sum of the number of isolated vertices, the smallest integers greater than or equal to half the size of each chain, and the and the largest integers less than or equal to half the size of each circuit.

4. Let $G = (X, E)$.

If $\forall x \in X deg(x) \leq 2$ then **MSIS3**(G) returns is the sum of the number of vertices of degree 0, the smallest integers greater than or equal to half the size of each chain, and the and the largest integers less than or equal to half the size of each circuit.

Otherwise pick $v \in X$ with $deg(c) > 2$. Then

$$\mathbf{MSIS3}(G) = \max(\{\mathbf{MSIS3}(G - v), \mathbf{MSIS3}(G - \text{Nbhd}(v)\} + 1).$$

5. 6. See the scanned diagrams (separate files).

7. We prove the claim by induction. First of all suppose that the chain has a single vertex (and so no edges). Then, by definition, $P(k; S_1) = k$.

Now suppose that for the chain S_n we have $P(k : S_n) = k(k-1)^{n-1}$. Consider the chain S_{n+1}. Apply the algo **ChromPoly**, choosing an edge at the end of the chain to delete or contract. We have $P(k : S_{n+1}) = P(k; S_n \cup \{pt\}) - P(k : S_n)$.

It is clear that if G is a graph and $\{pt\}$ is not in the set of vertices of G one has that, if $G \cup \{pt\}$ is the graph given by adding the point to the set of vertices, but no new edges, then the point can be give any of the k-colours to extend a colouring of G. Hence $P(k; G \cup \{pt\}) = kP(k; G)$.

Hence $P(k : S_{n+1}) = P(k; S_n \cup \{pt\}) - P(k : S_n) = k.k.(k-1)^{n-1} - k.(k-1)^{n-1} = k(k-1)^n$. Thus by the principle of induction we have that $P(k : S_n) = k.(k-1)^{n-1}$ for all n.

8. Let $G = (X, E)$. Let $c : X \longrightarrow \{0, 1, \ldots, \chi - 1\}$ witness that the chromatic number of G is χ. By the definition of being a (proper) colouring, for each $i \in \{0, 1, \ldots, \chi - 1\}$ and each pair of vertices x and y one has that if $c(x) = c(y) = i$ then $\{x, y\} \notin E$. Hence for each i we have that $\{x \in X \mid c(x) = i\}, = X_i$, say, is an independent set. So for each such i we have that $\mid X_i \mid \leq \alpha$. But $X_i \cap X_j = \emptyset$ when $i \neq j$, since c is a function, and $X = \bigcup_i X_i$. Thus $n = \mid X \mid = \Sigma_i \mid X_i \mid \leq \chi.\alpha$, as required.