Math 3502: Combinatorial Optimisation. Solutions 2.

1. We have the more general result that P is closed under complements. Given a language $L \in P$ decided by a TM M in time p, for a polynomial p. the language $\Sigma^* \setminus L(= \{x \in \Sigma^* \mid x \notin L\}$ is decided by a machine M' identical to M except that when M halts accepting (in q_Y) M' halts and rejects (in q'_N) and vice versa.

(In fact, by a theorem of Agrawal, Kayal and Saxena, $PRIME$ is in P.)

2. Suppose $A \leq_p B \leq_p C$. Let M show that A is reducible to B in polynomial time. So there is a polynomial $p(n)$ such that $T_M(n) \leq p(n)$ for all n, and M computes some $f : \Sigma^* \longrightarrow \Sigma^*$ such that $x \in A$ if and only if $f(x) \in B$.

Similarly let N show that B is reducible to C in polynomial time. So there is a polynomial $q(n)$ such that $T_N(n) \leq q(n)$ for all n, and N computes some $g : \Sigma^* \longrightarrow \Sigma^*$ such that $y \in B$ if and only if $g(y) \in C$.

We devise a new Turing machine K simply by first running M on input x and, instead of halting when M would have halted immediately running N. Thus K computes the function $g \circ f$. Since $|f(x)| \leq p(|x|)$ we have that $T_K(n) \leq q(p(n))$.

As p and q are polynomials we clearly have that $q(p(n))$ is a polynomial.

We also have that $x \in A$ if and only if $f(x) \in B$ if and only if $g(f(x)) \in C$. So K computes a reduction of A to C. Hence $A \leq_p C$.

3. This is a theorem due to König (1936).

Suppose that G is 2-colourable and that the (distinct) vertices $y_1, \ldots y_{2k+1}$ form an odd cycle. Suppose y_1 is coloured, without loss of generality, red. Then y_2 is coloured blue, y_3 is coloured red and so on, ultimately giving y_{2k+1} is coloured red. But this is a contradiction because y_1 and y_{2k+1} are adjacent.

For the converse colour G as follows.

Initialization: All vertices are colourless. Queue is empty.
Step 1: Pick a vertex x, colour it blue, put it in the queue.
Step 2: Remove the first vertex to be found in the queue, call it y.
Step 3: Find all the uncoloured neighbors of y, colour them the opposite colour of y, put them in the queue.
Step 4: If all the vertices are coloured stop, else go to Step 2.

If G is 2-colourable this will be a good colouring.

Let $d(x, v)$ be the length of the shortest chain between x and v. If x is colored blue, then for each red colored vertex u, we must have $d(x, u)$ odd, while for each blue colored vertex v, we will have $d(x, v)$ even.

Suppose that two neighbors, u and v, get the same color. Then d(x,u) and d(x,v) have the same parity (both odd or both even). The closed chain $x - u - v - x$ is therefore of odd length. If this closed chain is not simple, we can make it simple by removing edges, but this always reduces the number of edges by an even number. When we are finished "pruning" the chain, we will be left with a circuit of odd length.

4. Let the graph be $G = (X, E)$, where $X = \{v_1, \ldots, v_6\}$, where, for definiteness, the top row in the illustration consists of the vertices v_1, v_2 and v_3 and the bottom row is v_4, v_5 and v_6, both times going from left to right..

In order to reduce the question of whether the graph can be 2-coloured to an instance of SAT we use variables x_{im} with $1 \leq i \leq 6$ and $m \in \{1, 2\}$, where x_{im} is the truth value of the statement "vertex i gets colour m'. We then have the following clauses:

(•) $C_i = x_{i1}$ v x_{i2} for $1 \leq i \leq 6$. (Each vertex gets some colour.)

(•) $U_{ilm} = \neg x_{i1}$ v $\neg x_{i2}$ for $1 \leq i \leq 6$. (No vertex gets two colours.)

(•) $D_{ijm} = \neg x_{im}$ v $\neg x_{jm}$ for $1 \leq i, j \leq 6$ such that $\{v_i, v_j\} \in E$ and $1 \leq m \leq 2$. (The vertices of each edge in E have different colours.) Here

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}, \{v_4, v_5\}, \{v_5, v_6\}\}.$$

The set of clauses is satisfiable if and only if the graph is 2-colourable.

The set of clauses is satisfiable for $k = 2$, for example, if x_{11}, x_{31}, x_{51}, x_{22}, x_{42}, and x_{62} are assigned 'T' and x_{12}, x_{32}, x_{52}, x_{21}, x_{41}, and x_{61} are assigned 'F.'