

EdExcel Decision Mathematics 1

Linear Programming

Section 1: Formulating and solving graphically

Notes and Examples

These notes contain subsections on:

- [Formulating LP problems](#)
- [Solving LP problems](#)
- [Minimisation problems](#)
- [Integer programming](#)

Formulating LP problems

There are three important steps in formulating a linear programming problem:

- **Defining the variables**
It is essential to define your variables carefully. **Don't** just write something like "Let x be the energy drink and y be the refresher drink" – you won't get marks for this. It must be "Let x be the number of litres of energy drink produced, and y be the number of litres of refresher drink produced."
- **Finding the constraints**
Read the question carefully and make sure that you translate the constraints accurately. In this first section most constraints involve \leq , but this is not always the case!
- **Finding the objective function**
The objective function is the quantity that needs to be maximised (e.g. profit). In the next section you will also meet problems in which the objective function needs to be minimised (e.g. cost).

Solving LP problems

To solve a linear programming problem graphically, you need to ensure that you can draw an accurate graph and read off it.

The steps involved are as follows:

- **Draw the lines representing the constraints and shade out the unwanted region for each**
You should use graph paper for this!
It is essential that you can draw the graph of a straight line quickly and efficiently. Many constraint lines are of the form $ax + by = c$ (with a , b and c positive). The easiest method for these is to put first $x = 0$ and find

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the corresponding value of y , and then put $y = 0$ and find the corresponding value of x . You then have the two points where the line crosses the coordinate axes, which you can then join up.

For constraint lines in other forms, the graph may cross one of the axes at a negative point. Since you don't usually need negative parts of the graph, just find any two points on the line (not too near each other) and join them up.

You must draw the lines carefully, as inaccuracy will affect your solution.

For each constraint line you must then shade out the unwanted part. If you are not sure, choose a point (such as the origin for simplicity) and see if it satisfies the inequality. If it does, shade out the other side of the line.

Remember that usually $x \geq 0$ and $y \geq 0$, so if necessary shade out negative areas of the graph.



Try the **Inequality regions** activity, in which you match up sets of inequalities with the appropriate graph.

- **Find the coordinates of the vertices of the feasible region**

Once you have drawn in all your constraints and shaded the unwanted regions, you should be left with an unshaded area – the feasible region.

You need to find the coordinates of the vertices of this region. You can simply read these off the graph, as accurately as possible, and you will be given full credit in an exam for doing this, so long as your graph is accurate. Alternatively you can use simultaneous equations to find the exact coordinates of the points where the lines cross. However, this will probably take you longer.

The larger the numbers involved, the less accurate the solution from reading off a graph. Later, when you deal with integer programming problems, there are some situations where using simultaneous equations could save you some time.

- **Find the optimal solution**

Two methods are shown in the textbook. Either use the objective line method, in which you draw a line representing the objective function and move a ruler parallel to it until you reach the edge of the feasible region, or the vertex method, in which you work out the value of the objective function at each vertex. For the vertex method, you need to show that you have tried each of the vertices to be sure that you have found the maximum value (or minimum in some cases). Sometimes the optimal solution is on one of the coordinate axes – don't just assume that it is one near the middle!

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Make sure that you have given all the information required by the question, in terms of the original problem if appropriate, e.g. “The maximum income is £880, given by making 400 litres of the energy drink and 600 litres of the refresher drink” rather than just “maximum value = 880 at (400, 600)”.



There are a series of PowerPoint animations on the website which show each step of the process: **formulation** of the problem, **drawing the graph** of the problem, and solving the problem using either the **objective line method** or the **vertex method**.



You also will find a **Flash resource** on the website which demonstrates how the value of the objective function varies throughout the feasible region. You can change the objective function and the two constraints. The feasible region is shown on the graph. The black line represents the objective function – you can move it around, without going outside the feasible region, and find the maximum possible value of the objective function (given in the box at the bottom). When you press “Solve”, the vertices of the feasible region appear, and you can click on each one in turn to find the value of the objective function at each one.

Try to find a situation for which the vertex which gives the maximum possible value of the objective function is

- the one on the x -axis
- The one on the y -axis
- The one at the intersection of the two constraint lines



There is also an **Excel spreadsheet** that you can download from the website. This works in a similar way to the Flash resource, but you can have three constraints. You can move the line representing the objective function by changing the value of r in the equation $px + qy = r$.

You can use a graph plotting program such as Autograph (see **instructions**) to help you solve linear programming problems (although of course you must be able to draw the graphs accurately by hand for your examination).

Minimisation problems

Minimisation problems, whether they are linear programming or integer programming problems, are dealt with in a very similar way to maximisation problems. The constraints (or at least some of them) involve \geq rather than \leq , and you are of course looking for the smallest possible value of the objective function.

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Example 1

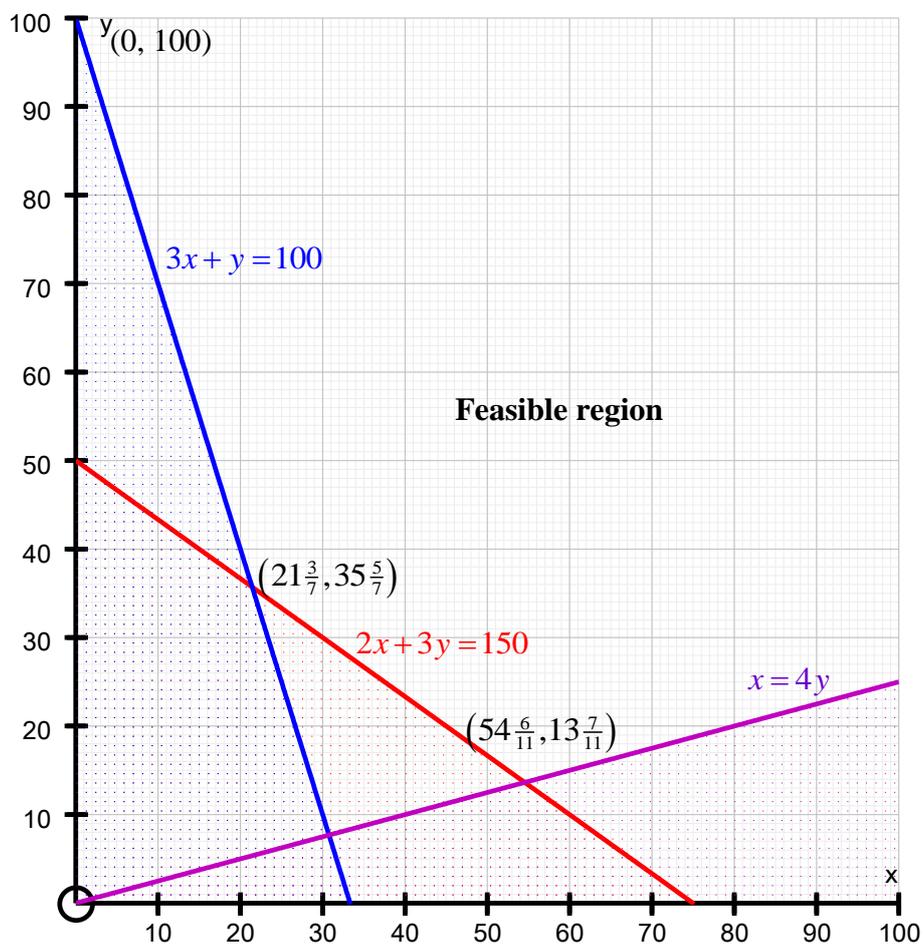
Solve the following linear programming problem

$$\begin{array}{ll} \text{Minimise} & 4x + 3y \\ \text{subject to} & 2x + 3y \geq 150 \\ & 3x + y \geq 100 \\ & x \leq 4y \end{array}$$



Solution

- (i) The graph of the feasible region is shown below. The coordinates of the vertices of the feasible region can be calculated exactly using simultaneous equations, or approximate values can be read off the graph.



$$\text{At } (0, 100), 4x + 3y = (4 \times 0) + (3 \times 100) = 300$$

$$\text{At } (21\frac{3}{7}, 35\frac{5}{7}), 4x + 3y = (4 \times 21\frac{3}{7}) + (3 \times 35\frac{5}{7}) = 192\frac{6}{7}$$

$$\text{At } (54\frac{6}{11}, 13\frac{7}{11}), 4x + 3y = (4 \times 54\frac{6}{11}) + (3 \times 13\frac{7}{11}) = 259\frac{1}{11}$$

The objective function has a minimum value of $192\frac{6}{7}$ when $x = 21\frac{3}{7}$ and $y = 35\frac{5}{7}$.

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Integer programming

Many problems involving maximising (or minimising) subject to certain constraints are in fact integer programming problems, in which the solution points must be integers, due to the practicalities of the situation. If the vertices of the feasible region happen to be integers, there is no problem, and you can just identify the vertex which gives the optimal solution, as for linear programming problems.

However, in many situations the vertices are not integers, and you then have to look at the integer points closest to the vertices of the feasible region. If the numbers involved are small, you may be able to read off the integer points nearest to the vertices, and check each one.



You can explore these ideas using the *Integer programming* Flash resource.

However, if the numbers are large, the graph may not give enough accuracy for this to be possible, and it is best to use simultaneous equations to find the exact coordinates of the vertices before finding the nearest integer points. Once you know the optimal vertex, you need to identify the integer points nearest to it. One approach is to take the two integer values of x either side of the exact value of the x -coordinate of the vertex, and finding the maximum possible value of y which is in the feasible region in each case. You can do this by substituting the integer value of x into each constraint and finding the maximum possible value of y for each constraint. You then need the smallest of these.

Example 2 shows how this is done.



Example 2

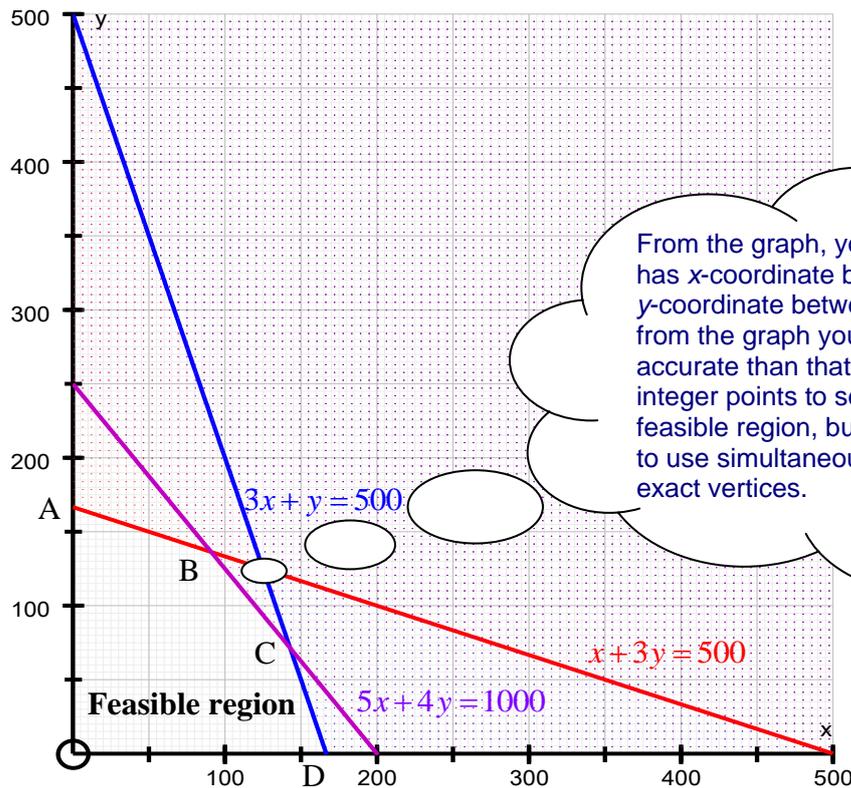
Solve the following integer programming problem.

$$\begin{array}{ll} \text{Maximise} & x + y \\ \text{subject to} & x + 3y \leq 500 \\ & 3x + y \leq 500 \\ & 5x + 4y \leq 1000 \end{array}$$

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Solution



From the graph, you can see that vertex B has x-coordinate between 90 and 95, and y-coordinate between 135 and 140, but from the graph you cannot really be more accurate than that. You could test various integer points to see if they are in the feasible region, but it is probably quicker to use simultaneous equations to find the exact vertices.

At vertex A: $3y = 500 \Rightarrow y = 166\frac{2}{3}$

At vertex B: $x + 3y = 500 \Rightarrow$

$$5x + 4y = 1000$$

$$5x + 15y = 2500$$

$$5x + 4y = 1000$$

$$11y = 1500 \Rightarrow y = 136\frac{4}{11}, \quad x = 90\frac{10}{11}$$

At vertex C: $3x + y = 500 \Rightarrow 12x + 4y = 2000$

$$5x + 4y = 1000$$

$$5x + 4y = 1000$$

$$7x = 1000 \Rightarrow x = 142\frac{6}{7} \quad y = 71\frac{3}{7}$$

At vertex D: $3x = 500 \Rightarrow x = 166\frac{2}{3}$

At $(0, 166\frac{2}{3})$, $x + y = 166\frac{2}{3}$

At $(90\frac{10}{11}, 136\frac{4}{11})$, $x + y = 227\frac{3}{11}$

At $(142\frac{6}{7}, 71\frac{3}{7})$, $x + y = 214\frac{2}{7}$

At $(166\frac{2}{3}, 0)$, $x + y = 166\frac{2}{3}$

You need to know which vertex is the optimal one, so that you know where to look for integer points.

Look for integer points near $(90\frac{10}{11}, 136\frac{4}{11})$:

When $x = 90$, the maximum possible integer value of y in the feasible region is 136.

Substitute $x = 90$ into each constraint and find the largest possible value of y to satisfy them all.

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When $x = 91$, the maximum possible integer value of y in the feasible region is 136.

Substitute $x = 91$ into each constraint and find the largest possible value of y to satisfy them all.

The objective function is therefore maximised when $x = 91$ and $y = 136$. The maximum value of the objective function is 227.

The next example shows a situation for which it is easy to pick out integer points close to the vertices directly from the graph, so there is no need to use simultaneous equations. However, in this case (unusually) the optimal point is not the closest point to one of the vertices.



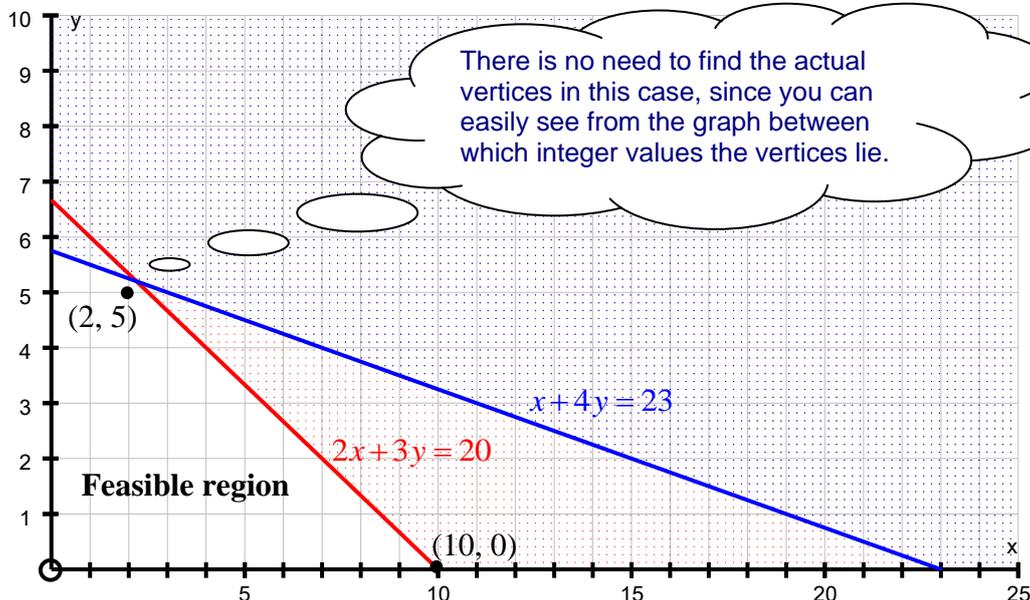
Example 3

Solve the following integer programming problem.

$$\begin{array}{ll} \text{Maximise} & 5x + 8y \\ \text{subject to} & 2x + 3y \leq 20 \\ & x + 4y \leq 23 \end{array}$$

Solution

The feasible region is shown on the graph below.



The point $(10, 0)$ is an integer point and is also a vertex.

The point $(2, 5)$ is clearly the integer point nearest to the middle vertex.

The nearest integer point to the vertex on the y -axis is $(0, 5)$, but there is no point trying that since $(2, 5)$ will clearly give a higher objective function.

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At (2, 5), $5x + 8y = (5 \times 2) + (8 \times 5) = 50$

At (10, 0), $5x + 8y = (5 \times 10) + (8 \times 0) = 50$

However, in this case there are integer points which gives a higher value for the objective function.

At (7, 2), $5x + 8y = (5 \times 7) + (8 \times 2) = 51$

At (4, 4), $5x + 8y = (5 \times 4) + (8 \times 4) = 52$

The maximum value of the objective function is 52, when $x = 4$ and $y = 4$.



Example 3 illustrates an unusual situation in which the optimal point is not the closest point to either vertex. You can see why this occurs using the **Integer Programming Flash resource**. Set up the constraints and objective function as for Example 2. Move the line corresponding to the objective function gradually upwards and you will see that it passes (2, 5) and (10, 0) before (7, 2) and lastly (4, 4) – although at this point it has still not passed the actual vertex of the graph.

Notice that in this example the gradient of the objective function line is very close to the gradient of one of the constraint lines, and the points (7, 2) and (4, 4) lie on the boundary of the feasible region.

You should be aware of situations like this, but they are unusual, and in most cases it is sufficient to look at integer solutions near the vertices. In an examination you would probably be given a hint.

Example 4 is an integer programming problem involving minimisation. It is the same problem as in Example 1, but an integer solution is required.



Example 4

Solve the following integer programming problem

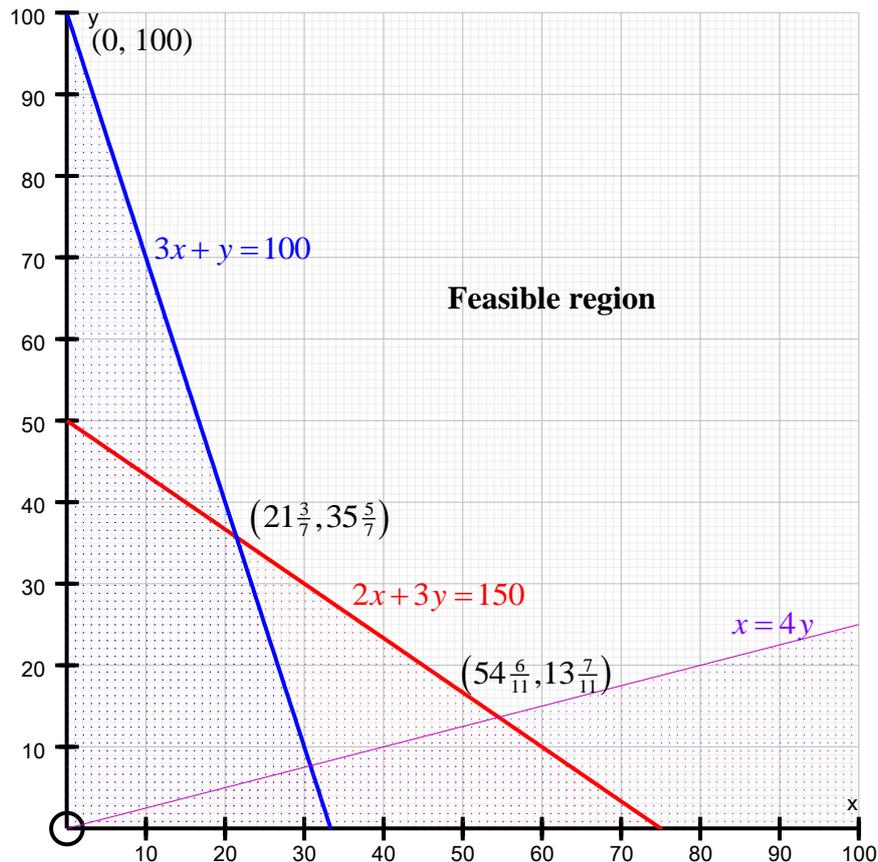
$$\begin{array}{ll} \text{Minimise} & 4x + 3y \\ \text{subject to} & 2x + 3y \geq 150 \\ & 3x + y \geq 100 \\ & x \leq 4y \end{array}$$



Solution

- (i) The graph of the feasible region is shown below. The coordinates of the vertices of the feasible region can be calculated exactly using simultaneous equations, or approximate values can be read off the graph.

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At $(0, 100)$, $4x + 3y = (4 \times 0) + (3 \times 100) = 300$

At $(21\frac{3}{7}, 35\frac{5}{7})$, $4x + 3y = (4 \times 21\frac{3}{7}) + (3 \times 35\frac{5}{7}) = 192\frac{6}{7}$

At $(54\frac{6}{11}, 13\frac{7}{11})$, $4x + 3y = (4 \times 54\frac{6}{11}) + (3 \times 13\frac{7}{11}) = 259\frac{1}{11}$

The objective function has a minimum value of $192\frac{6}{7}$ when $x = 21\frac{3}{7}$ and $y = 35\frac{5}{7}$.

- (ii) Look at integer points close to $(21\frac{3}{7}, 35\frac{5}{7})$.

When $x = 21$, the minimum possible value of y in the feasible region is 37

At $(21, 37)$, $4x + 3y = (4 \times 21) + (3 \times 37) = 195$

Substitute $x = 21$ into each constraint and find the smallest possible value of y to satisfy them all.

When $x = 22$, the minimum possible value of y in the feasible region is 36

At $(22, 36)$, $4x + 3y = (4 \times 22) + (3 \times 36) = 196$

Substitute $x = 22$ into each constraint and find the smallest possible value of y to satisfy them all.

For the integer programming problem, the objective function has a minimum value of 195 when $x = 21$ and $y = 37$.

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In Example 2, the maximum value for the linear programming problem was $227\frac{3}{11}$ and the maximum value for the integer programming problem was 227, so you could feel confident that you had the optimal integer point. In Example 4, however, the minimum value for the linear programming problem was $192\frac{6}{7}$, and the minimum value for the integer programming problem was 195. This might make you think that you should look for a point which will give you 193, but that is not in fact possible in this case.



You may like to try the **Linear Programming Matching activity**, in which you match up linear programming problems with their graphs and solutions.