TENSOR PRODUCTS, RESTRICTION AND INDUCTION.

ANDREI YAFAEV

Our first aim in this chapter is to give meaning to the notion of **product** of characters.

Let V and W be two finite dimensional vector spaces over $\mathbb{C}$ with bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ repectively. Define a **symbol** $v_i \otimes w_j$. The tensor product space $V \otimes W$ is the mn-vector spacewith basis

$$\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

The symbol $v_i \otimes w_j$ is bilinear.

In general, let $v = \sum \lambda_i v_i$ and $w = \sum \mu_j w_j$, then

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j)$$

For example

$$(2v_1 - v_2) \otimes (w_1 + w_2) = 2v_1 \otimes w_1 + 2v_1 \otimes w_2 - v_2 \otimes w_2 - v_2 \otimes w_2$$

In other words, to calculate with tensor products, just use the bilinearity. **Caution** Not every tensor can be expressed as $v \otimes w$, indeed $v_1 \otimes w_1 + v_2 \otimes w_2$ can not be expressed in this form.

Proposition 0.1. *Let $e_1, \dots, e_m$ be a basis of V and $f_1, \dots, f_n$ a basis of W. Then*

$$\{e_i \otimes f_j\}$$

is a basis of $V \otimes W$.

Proof. It is obvious that these elements generate $V \otimes W$ (by bilinearity) and there are mn of them, hence it is a basis. □

Suppose now that V and W are $\mathbb{C}[G]$-modules. One defines the structure of $\mathbb{C}[G]$-module on $V \otimes W$ by

$$g(v_i \otimes w_j) = (gv_i) \otimes (gw_j)$$

By bilinearity we obtain

$$g(v \otimes w) = (gv) \otimes (gw)$$

It is obvious that this gives $V \otimes W$ a structure of $\mathbb{C}[G]$-module.

Proposition 0.2. *Let V and W be $\mathbb{C}[G]$-modules with characters χ and ψ. The character ϕ of $V \otimes W$, is the product $\chi\psi$:*

$$\chi\psi(g) = \chi(g)\psi(g)$$

Proof. Let $g \in G$. We can diagonalise its action on V and W. Hence there exist bases $\{e_i\}$ of V and $\{f_i\}$ of W such that

$$ge_i = \lambda_i e_i \text{ and } gf_j = \mu_j f_j$$

Then

$$\chi(g) = \sum \lambda_i \text{ and } \psi(g) = \sum \mu_j$$

We obtain

$$g(e_i \otimes f_j) = (ge_i) \otimes (gf_j) = \lambda_i \mu_j (e_i \otimes f_j)$$

As $\{e_i \otimes f_j\}$ forms a basis of $V \otimes W$, we obtain

$$\phi(g) = \sum_{i,j} \lambda_i \mu_j = \chi(g)\psi(g)$$

□

This gives meaning to the **product of two characters**, indeed the consequence of this proposition is :

Corollary 0.3. *The product of two characters is a character.*

Take the character table of S_4.

g_i	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1
$\chi_3\chi_4$	6	0	0	-2	0
χ_4^2	9	1	0	1	1

We see that

$$\chi_3\chi_4 = \chi_4 + \chi_5; \quad \chi_4^2 = \chi_1 + \chi_3 + \chi_4 + \chi_5$$

We show the following:

Proposition 0.4. *Let χ be a character of G and let λ be a* **linear** *character (recall that it means that the degree of λ is one). Suppose χ is irreducible, then $\lambda\chi$ is irreducible.*

Proof. For any g, $lambda(g)$ is a root of unity, therefore $\lambda(g)\overline{\lambda(g)} = 1$.

We calculate:

$$< \lambda\chi, \lambda\chi >= \frac{1}{|G|}\sum_g \chi(g)\lambda(g)\overline{\chi(g)\lambda(g)} =< \chi, \chi >= 1$$

hence $\lambda\chi$ is irreducible. □

We will now see how to decompose the character χ^2 and apply it to character tables of symmetric groups.

Let V be a $\mathbb{C}[G]$-module with character χ, the module $V \otimes V$ has character χ^2. Define the linear transformation

$$T(v_i \otimes v_j) = v_j \otimes v_i$$

Then for all v, w, we have $T(v \otimes w) = w \otimes v$.

Let

$$S(V\otimes V) = \{x \in V\otimes V : T(x) = x\}, \quad A(V\otimes V) = \{x \in V\otimes V : T(x) = -x\}$$

called the **symmetric** and **antisymmetric** part of $V \otimes V$.

The spaces $S(V \otimes V)$ and $A(V \otimes V)$ are $\mathbb{C}[G]$-submodules and

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V)$$

It is obvious that T is a $\mathbb{C}[G]$-homomorphism and hence for $x \in S(V \otimes V)$ and g in G,

$$T(gx) = gT(x) = gx$$

hence $gx \in S(V \otimes V)$. Similarly, $A(V \otimes V)$ is a $\mathbb{C}[G]$ submodule.

For the direct sum:

$x \in S(V \otimes V) \cap A(V \otimes V)$, then $x = T(x) = -x$ hence $x = 0$.

And for any $x \in V \otimes V$, we have

$$x = \frac{1}{2}(x + T(x)) + \frac{1}{2}(x - T(x))$$

As T^2 is the identity, we see that $\frac{1}{2}(x + T(x)) \in S(V \otimes V)$ and $\frac{1}{2}(x - T(x)) \in A(V \otimes V)$.

Note that the symmetric part contains all the tensors $v\otimes w+w\otimes v$ and antisymmetric part - all the tensors $v\otimes w-w\otimes v$. In fact $v_i\otimes v_j+v_j\otimes v_i$ ($i \leq j$) form a basis of S and $v_i \otimes v_j - v_j \otimes v_i$ ($i < j$) form a basis of A. The dimenasion of S is $\frac{n(n+1)}{2}$ and that of A is $\frac{n(n-1)}{2}$.

Proposition 0.5. *Write*

$$\chi^2 = \chi_S + \chi_A$$

then,

$$\chi_S(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)) \text{ and } \chi_A(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$$

Proof. As usual, choose a basis e_i of V such that $ge_i = \lambda_i e_i$. Then

$$g(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

It follows that

$$\chi_A(g) = \sum_{i<j} \lambda_i \lambda_j$$

Now, $g^2 e_i = \lambda_i^2 e_i$, therefore $\chi(g) = \sum \lambda_i$ and $\chi(g^2) = \lambda_i^2$. It follows that

$$\chi(g)^2 = (\chi(g))^2 = \sum \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j = \chi(g^2) + 2\chi_A(g)$$

hence

$$\chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$$

and the equality for χ_S follows from $\chi^2(g) = \chi_A(g) + \chi_S(g)$. □

Take the character table of S_4 above, we get:

$$\chi_S = \chi_1 + \chi_3 + \chi_4; \quad \chi_A = \chi_5$$

1. Character table of S_5.

The group S_5 has 7 conjugacy classes, as follows

g_i	$g_1 = 1$	$g_2 = (12)$	$g_3 = (123)$	$g_4 = (12)(34)$	$g_5 = (1234)$	$g_6 = (123)(45)$	(1234
$\lvert C_G(g_i)\rvert$	120	12	6	8	4	6	5

The group S_5 has exactly two irreducible characters of degree one : χ_1 (trivial) and χ_2 (sign).

In fact any symmetric group has exactly two irreducible linear characters : the trivial and the sign. This is a consequence of the fact that A_n is the derived subgroup of S_n, hence any homomorphism from S_n into a commutative group factors through A_n. The quotient S_n/A_n is of order two hence the non-trivial element is sent either to 1 or -1. This gives exactly two linear characters.

Here is a generality on the **permutation character** of S_n.

Let $G = S_n$ be the symmetric group. It has a natural n-dimensional representation defined by

$$ge_i = e_{gi}$$

(the permutation representation). Let π be its character.

For $g \in G$, define

$$fix(g) = \{i : 1 \leq i \leq n \text{ and } gi = i\}$$

Then

$$\pi(g) = |fix(g)|$$

Proposition 1.1. *Let G be a subgroup of S_n, let $\mu\colon G \longrightarrow \mathbb{C}$ be the function defined by*

$$\mu(g) = |fix(g)| - 1$$

Then μ is a character of G.

Proof. The permutation representation V always has an invariant subspace which is

$$U = Span(u_1 + u_2 + \cdots + u_n)$$

By Mashke's theorem it has a complement W, a $\mathbb{C}[G]$-submodule such that

$$V = U \oplus W$$

Let μ be the character of W, then

$$\pi = 1_G + \mu$$

where $1_G(g) = 1$ for all g. We then have

$$\mu(g) = |fix(g)| - 1$$

□

Going back to S_5, we let χ_3 be the permutation character.

Let us determine the values of χ_3.

(1) $|fix(1)| = 5$ hence $\chi_3(1) = 4$
(2) $|fix(1,2)| = 3$ hence $\chi_3((1,2)) = 2$
(3) $|fix(1,2,3)| = 2$ hence $\chi_3((1,2,3)) = 1$
(4) $|fix(1,2)(3,4)| = 1$ hence $\chi_3((1,2)(3,4)) = 0$
(5) $|fix(1,2,3,4)| = 1$ hence $\chi_3((1,2,3,4)) = 0$
(6) $|fix(1,2,3)(4,5)| = 0$ hence $\chi_3((1,2,3)(4,5)) = -1$
(7) $|fix(1,2,3,4,5)| = 0$ hence $\chi_3((1,2,3,4,5)) = -1$

The values of χ_3 are as follows $4, 2, 1, 0, 0, -1, -1$. We calculate

$$< \chi_3, \chi_3 >= 4^2/120 + 2^2/12 + 1^2/6 + (-1)^2/6 + (-1)^2/5 = 1$$

It follows that **the character χ_3 is irreducible**.

Now, $\chi_4 = \chi_3\chi_2$ is also irreeducible.

We already have 4 irreducible characters.

Need three more...

Consider $\chi_3^2 = \chi_S + \chi_A$.

We have

$$\chi_S(g) = \frac{1}{2}(\chi_3(g)^2 + \chi_3(g^2))$$

and

$$\chi_A(g) = \frac{1}{2}(\chi_3(g)^2 - \chi_3(g^2))$$

To calculate values of χ_S and χ_A, we calculate: $1^2 = 1$, $g_2^2 = 1$, $g_3^2 \sim g_3$, $g_4^2 = 1$, $g_5^2 \sim g_4$, $g_6^2 \sim g_3$, $g_7^2 \sim g_7$.

We find $\chi_S : 10, 4, 1, 2, 0, 1, 0$ and $\chi_A : 6, 0, 0, -2, 0, 0, 1$. Call it χ_5.

One caclulates : $< \chi_A, \chi_A >= 1$ hence χ_A **is a new irreducible character**.

Notice here that $\chi_2\chi_A = \chi_A$ hence multiplying by χ_2 **does not give a new character**.

Now look at χ_S. We have $< \chi_S, \chi_S >= 3$ hance χ_S is a sum of **three** irreducible characters.

Next :

$$< \chi_S, \chi_1 >= 10/120 + 4/12 + 1/6 + 2/8 + 1/6 = 1,$$

$$< \chi_S, \chi_3 >= 40/120 + 8/12 + 1/6 - 1/6 = 1,$$

$$< \chi_S, \chi_S >= 100/120 + 16/12 + 1/6 + 4/8 + 1/6 = 3$$

Write $\chi_S = \sum \lambda_i \chi_i$, we have $\sum \lambda_i^2 = 3$ hence exactly three λ_is are equal to 1 and we already have $\lambda_1 = \lambda_3 = 1$.

Therefore

$$\chi_S = \chi_1 + \chi_3 + \psi$$

where ψ is some irreducible character.

We have

$$\chi_S(1) = \chi_1(1) + \chi_3(1) + \psi(1) = \frac{1}{2}(\chi_3(1)^2 + \chi_3(1)) = \frac{1}{2}(16 + 4) = 10$$

As $\chi_1(1) = 1$ and $\chi_3(1) = 4$, we find that $\psi(1) = 5$.

Hence ψ is a new irreducible character, we let $\chi_6 = \psi$. Using

$$\chi_6(g) = \chi$$

We find

$$\chi_6 : 5, 1, -1, 1, -1, 1, 0$$

Finally, $\chi_7 = \chi_6\chi_2$ is the last character.

We get the complete character table for S_5:

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

Notice that all entries are integers !

2. Character table of S_6.

The group S_6 is of order 720.
It has 11 conjugugacy classes.
We denote them by their shape :

$$g_1 = 1, g_2 = (2), g_3 = (3), g_4 = (2,2), g_5 = (4), g_6 = (3,2), g_7 = (5)$$

$$g_8 = (2,2,2), g_9 = (3,3), g_{10} = (4,2), g_{11} = (6)$$

The sizes of centralisers are $720, 48, 18, 16, 8, 6, 5, 48, 18, 8, 6$.

As before we have two linear characters χ_1 and χ_2.

Next, as before, consider the permutation character : $\chi_3(g) = |fix(g)| - 1$, the values of χ_3 are $5, 3, 2, 1, 1, 0, 0, -1, -1, -1, -1$ and one calculates

$$< \chi_3, \chi_3 >= 1$$

We get another irreducible character by setting $\chi_4 = \chi_2 \chi_3$.

Next, as before we consider

$$\chi_3^2 = \chi_S + \chi_A$$

We have

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
χ_3	5	3	2	1	1	0	0	-1	-1	-1	-1
χ_S	15	7	3	3	1	1	0	3	0	1	0
χ_A	10	2	1	-2	0	-1	0	-2	1	0	1

One finds that $< \chi_A, \chi_A >= 1$. We let $\chi_5 = \chi_A$, this is the new irreducible character.

In this case (unlike in the case of S_5), $\chi_2 \chi_5 = \chi_6$ is a new irreducible character.

Finally, we calculate:

$$< \chi_S, \chi_S >= 3, < \chi_S, \chi_1 >= 1, < \chi_S, \chi_3 >= 1$$

hence, as before there is an irreducible character ψ such that

$$\chi_S = \chi_1 + \chi_3 + \psi$$

This gives χ_7 of degree 9 and $\chi_8 = \chi_2\chi_7$ is another irreducible character. The table so far is as follows:

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1	-1	1	1	-1
χ_3	5	3	2	1	1	0	0	-1	-1	-1	-1
χ_4	5	-3	2	1	-1	0	0	1	-1	-1	1
χ_5	10	2	1	-2	0	-1	0	-2	1	0	1
χ_6	10	-2	1	-2	0	-1	0	2	1	0	-1
χ_7	9	3	0	1	-1	0	-1	3	0	1	0
χ_8	9	-3	0	1	1	0	-1	-3	0	1	0

We will recover the three remining characters from orthogonality relations.

Let s be the permutation $(1,2)$ and t the permutation $(1,2)(3,4)$, these are elements of order two. It is a general fact that if g has order two, then $\chi(g)$ is an integer. Indeed, $\chi(g)$ is a sum of square roots of one, they are ± 1.

Let χ_9, $\chi_1 0$ and χ_{11} be the three remaining characters.

Column orthogonality gives:

$$\sum_{i=1}^{11} \chi_i(s) = 48 = |C_G(s)|$$

Hence

$$\chi_9(s)^2 + \chi_1 0(s)^2 + \chi_{11}(s)^2 = 2$$

By reodering the characters, we assume that

$$\chi_9(s)^2 = \chi_1 0(s)^2 = 1 \text{ and } \chi_{11}(s)^2 = 0$$

Now, the character $\chi_2\chi_9$ is an irreducible character not equal to any of the $\chi_1, \dots, \chi_8$ (because they come in pairs !)

By definition of χ_2, we have

$$\chi_2\chi_9(s) = \chi_2(s)\chi_9(s) = -\chi_9(s)$$

As $\chi_9(s) = \pm 1$, we see that $\chi_2\chi_9 \neq \chi 9$ and can not be equal to χ_{11} ($\chi_{11}(s) = 0$) hence

$$\chi_2\chi_9 = \chi_{10}$$

After, if necessary, renumbering the characters, we have

$$\chi_9(s) = 1, \chi_{10}(s) = -1$$

We have completely determined the values of χ_is at s. Now we have the table

g_i	1	s	t
χ_9	a	1	d
χ_{10}	b	-1	e
χ_{11}	c	0	f

Write orthogonality relations:

$$\sum \chi_i(1)\chi_i(s) = 0 \quad \sum \chi_i(s)\chi_i(t) = 0$$

$$\sum \chi_i(t)\chi_i(t) = 16 \quad \sum \chi_i(1)\chi_i(t) = 0$$

This gives

$$a - b = 0 \quad d - e = 0$$

$$d^2 + e^2 + f^2 = 2 \quad ad + be + cf = 10$$

and it is easy to see that the only solutions in integers are

$$d = e = 1 \quad f = 0 \quad a = b = 5$$

Finally, using $\sum_i \chi_i(1)^2 = 720$ gives $c = 16$.

The rest of the table is determined by column orthogonality...

3. Restriction and induction.

Let H be a subgroup of G. Then $\mathbb{C}[H] \subset \mathbb{C}[G]$ and any $\mathbb{C}[G]$-module V can be viewed as a $\mathbb{C}[H]$-module. This is called the restriction from G to H and we denote this module

$$V \downarrow H$$

Let χ be the character of V. The character of $V \downarrow H$ is obtained from χ by evaluating it on elements of H only, we denote it $\chi \downarrow H$. We call it the restriction of χ to H. Viewing χ as a function from G to $\mathbb{C}$, $\chi \downarrow H$ is simply the restriction of this function to H.

The inner product of characters of G, $<,>_G$ yields, by restriction the inner product $<,>_H$ of characters of H. If χ is a character of G and ψ_i are irreducible characters of H, we have

$$\chi \downarrow H = d_1\psi_i + \cdots + d_r\psi_r$$

and we have

$$d_i =< \chi \downarrow H, \psi_i >_H$$

They satisfy the following

Proposition 3.1. *Let χ be an irreducible character of G and $\psi_1, \dots, \psi_r$ irreducible characters of H. Then*

$$\chi \downarrow H = d_1\psi_1 + \cdots + d_r\psi_r$$

where the d_i satisfy

$$\sum d_i^2 \leq |G : H|$$

with equality if and only if $\chi(g) = 0$ **for all** $g \in G\backslash H$.

Proof. We have

$$\sum d_i^2 =< \chi \downarrow H, \chi \downarrow H >_H = \frac{1}{|H|}\sum_{h\in H} \chi(h)\overline{\chi(h)}$$

As χ is an irreducible character, we have

$$1 =< \chi, \chi >_G = \frac{1}{|G|}\sum_{h\in H} \chi(h)\overline{\chi(h)} + K = \frac{|H|}{|G|}\sum d_i^2 + K$$

with $K = \frac{1}{|G|}\sum_{g\notin H} \chi(g)\overline{\chi(g)}$. Of course $K \geq 0$ and $K = 0$ if and only if $\chi(g) = 0$ for all $g \notin H$. □

We have the following:

Proposition 3.2. *Let H be a subgroup of G and ψ a non-zero character of H. There exists an irreducible character χ of G such that*

$$< \chi \downarrow H, \psi > \neq 0$$

Proof. Let χ_i be irreducible characters of G and let χ_{reg} be the regular character. We have

$$\chi_{reg} = \sum_{i=1}^{r} \chi_i(1)\chi_i$$

Now

$$0 \neq \frac{|G|}{|H|}\psi(1) =< \chi_{reg} \downarrow H, \psi >_H = \sum \chi_i(1) < \chi_i \downarrow H, \psi >_H$$

(the first equality here comes from the fact that $\chi_{reg}(1) = |G|$ and zero otherwise).

It follows that some $< \chi_i \downarrow H, \psi > \neq 0$. □

We can obtain more information when H **is a normal subgroup of** G.

Theorem 3.3 (Clifford's theorem)**.** *Suppose H is a normal subgroup of G and let χ be an irreducible character of G. Write*

$$\chi \downarrow H = d_1\psi_1 + \cdots + d_r\psi_r$$

Then

(1) *The ψ_is all have the same degree.*
(2)
$$\chi \downarrow H = e(\psi_1 + \cdots + \psi_m)$$

Proof. Let V be a $\mathbb{C}[G]$-module with character χ (necessarily irreducible) and U an irreducible $\mathbb{C}[H]$-submodule of $V \downarrow H$. For $g \in G$, let
$$gU = \{gv : v \in U\}$$
As H is a normal subgroup of G ($gHg^{-1} = H$), gU is a $\mathbb{C}[H]$-submodule of $V \downarrow H$. If W is a $\mathbb{C}[H]$-submodule of gU, then $g^{-1}W$ is a $\mathbb{C}[H]$-submodule of V. Now, U is ireducible, hence $W = \{0\}$ or $W = gU$. It follows that gU **is an irreducible submodule of** $V \downarrow H$**.**

Of course all gU have the same dimension. We have a direct sum decomposition:
$$V \downarrow H = \oplus_{g \in G} gU$$
(the sum is direct because modules are simple) and ψ_is are characters of some of the gUs which all have the same dimension (equal to d_i). This proves the first claim.

For the second, let
$$e =< \chi \downarrow H, \psi_1 >$$
and let X_1 be the submodule of $V \downarrow H$ whose chaacter is $e\psi_1$. Then
$$X_1 = U_1 \oplus \cdots \oplus U_e$$
where each U_e has character ψ_1.

Now, for any g in G, gX_1 is a direct sum of isomorphic $\mathbb{C}[H]$-modules gU_i

We just need an argument to show that they are isomorphic. We have $U_i \cong U_j$ and we need to show that $gU_i \cong gU_j$. If $\phi \colon U_i \longrightarrow U_j$ is an isomorphism of $\mathbb{C}[H]$-modules, then $\theta \colon gU_i \longrightarrow gU_j$ defined by $\theta(gu) = g\phi(u)$. Verifications that this is a $\mathbb{C}[H]$ morphism (using the fact that H is normal) are left to the reader.

The module $V \downarrow H$ is a sum of the gX_1. We write
$$V \downarrow H = X_1 \oplus \cdots \oplus X_m$$
where X_is are gX_1 for some $g \in G$ and pairwise non-isomorphic.

It follows that
$$\chi \downarrow H = e(\psi_1 + \cdots + \psi_m)$$
□

Suppose now that the index of H in G is two. We will typically be interested in $A_n \subset S_n$. Then for any irreducible character χ of G, either $\chi \downarrow H$ is irreducible or $\chi \downarrow H$ is a sum of two irreducible characters of the same degree.

To see this, write

$$\chi \downarrow H = d_1\psi_1 + \cdots + d_r\psi_r$$

where $\sum_i d_i^2 \leq 2$. Hence d_is are either $1, 1$ or 1.

As we have $G/H \cong C_2$, we can define a character λ of G by

$$\lambda(g) = 1 \text{ if } g \in H$$

and

$$\lambda(g) = -1 \text{ if } g \notin H$$

In the case $G = S_n$ and $H = A_n$, this is simply the sign.

Now, for irreducible characters χ of G, χ and $\lambda\chi$ are irreducible of the same degree. We have

Proposition 3.4. *The following are equivalent*

1. $\chi \downarrow H$ *is irreducible*
2. $\chi(g) \neq 0$ *for some* $g \in G$ *with* $g \notin H$
3. *The characters* χ *and* $\lambda\chi$ *are not equal.*

We have seen that $\sum d_i < 2$ (strict inequality)if and only if $\chi(g) \neq 0$ for some $g \in G$ and $g \notin H$. The inequality is strict precisely when $\sum d_i^2 < 2$.

Also $\lambda\chi(g) = \chi(g)$ if $g \in H$ and $-\chi(g)$ is $g \notin H$. So $\chi(g) \neq 0$ for $g \notin H$ if and only if $\lambda\chi \neq \chi$.

Proposition 3.5. *Suppose that* H *is a normal subgroup of index* 2 *in* G *and that |chi is an irreducible character of* G *such that* $\chi \downarrow H$ *is irreducible.*

If ϕ *is an irreducible character of* G *which satisfies*

$$\phi \downarrow H = \chi \downarrow H$$

then either $\phi = \chi$ *or* $\phi = \chi\lambda$.

Proof. We have

$$(\chi + \lambda\chi) = 2\chi(g) \text{ if } g \in H \text{ and } 0 \text{ otherwise}$$

Therefore

$$< \chi + \lambda\chi, \phi >= \frac{1}{|G|}\sum_{g \in H} 2\chi(g)\overline{\phi(g)} = \frac{1}{H}\sum_{g \in H} \chi(g)\overline{\phi(g)}$$

But $< \chi \downarrow H, \phi \downarrow H >= 1$ and $\phi \downarrow H = \chi \downarrow H$ hence $< \chi + \lambda\chi, \phi >= 1$ which implies that either $\phi = \chi$ or $\phi = \lambda\chi$. $\square$

Finaly we analyse the case where the character $\chi \downarrow H$ is reducible.

Proposition 3.6. *Suppose that H is a normal subgroup of index two of G and χ an irreducible character of G for which $\chi \downarrow H$ is the sum of two irreducible characters $\chi \downarrow H = \psi_1 + \psi_2$. If ϕ is a character such that $\phi \downarrow H$ has ψ_1 or ψ_2 in its decompositiiton, then $\phi = \chi$.*

Proof. We have $\chi(g) = 0$ for $g \notin H$, therefore

$$< \phi, \chi >_G = \frac{1}{2} < \phi \downarrow H, \chi \downarrow H >_H$$

if $\phi \downarrow H$ has ψ_1 or ψ_2 as constituent, then $< \phi \downarrow H, \chi \downarrow H >_H \neq 0$ hence $< \phi, \chi >_G \neq 0$ in which case it must be one. Therefore $\phi = \chi$ (χ is irreducible !). □

To summarise:
Suppose G is a finite group and H a subgroup of index 2.

(1) Each irreducible character χ of G **non-zero somewhere outside of** H restricts to an irreducible character of H.
 These characters come in pairs χ and $\lambda\chi$, they restrict to the same character on H.
(2) If χ irreducible on G **is zero everywhere outside** H, then χ restricts to the sum of two distinct irreducible characters **of same degree**.
 These two characters come from no other irreducible character of G.
(3) Every irreducible character appears among those obtained by restricting irreducible characters of G.
 Let ψ be an irreducible character of H. There exists χ irreducible of G such that $< \chi \downarrow H, \psi > \neq 0$. Now, either $\chi \downarrow H$ is irreducible in which case $\chi \downarrow H = \psi$ (necessarily $< \chi \downarrow H, \psi >= 1$) or $\chi \downarrow H = \psi_1 + \psi_2$ in which case we see that ψ is ψ_1 or ψ_2.

Let us apply what preceeded to A_5.

The group $H = A_5$ is of order 60 and is a normal subgroup of index 2 in S_5. It has 5 conjugacy classes $g_1 = 1$, $g_2 = (123)$, $g_3 = (1,2)(3,4)$, $g_4 = (12345)$ and $g_5 = (13452)$ and centraliserz have sizes 60, 3, 4, 5 and 5.

Look at our 7 characters $\chi_1, \dots, \chi_7$ of S_5 and the table we constructed previously. We see that χ_1, χ_3 and χ_6 are nonzero somewhere outside of A_4. Therefore $\chi_1 \downarrow H = \psi_1$, $\chi_3 \downarrow H = \psi_3$ and $\chi_6 \downarrow H = \psi_6$ give three irreducible characters of H.

Notice that $\chi_5(g) = 0$ for $g \notin H$. Hence $\chi_5 \downarrow H = \psi_4 + \psi_5$ where ψ_4 and ψ_5 are distinct irreducible characters of H and they are of degree three (they have the same degree and $\psi_4(1) + \psi_5(1) = \chi_5(1) = 6$).

Because χ_2 and χ_1 restrict to the same character; χ_4, χ_3 restrict to the same character and χ_7, χ_6 restrict to the same character, we see that $\psi_1, \ldots, \psi_5$ are distinct irreducible characters of A_5 and **this is a complete list**.

We have the table:

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5
ψ_1	1	1	1	1	1
ψ_2	4	1	0	-1	-1
ψ_3	5	-1	1	0	0
ψ_4	3	α_2	α_3	α_4	α_5
ψ_5	3	β_2	β_3	β_4	β_5

We will recover α_is and β_is by column orthogonality.

Because $\chi_5 \downarrow H = \psi_4 + \psi_5$, we have

$$\alpha_2 + \alpha_2 = 0, \quad \alpha_3 + \alpha_3 = -2, \quad \alpha_4 + \beta_4 = \alpha_5 + \beta_5 = 1$$

By column orthogonality, we get

$$3 = 3 + \alpha_1^2 + \alpha_2^2$$
$$4 = 2 + \alpha_3^2 + \beta_3^2$$
$$5 = 2 + \alpha_4^2 + \beta_4^2 = 2 + \alpha_5^2 + \beta_5^2$$

This immediately gives $\alpha_2 = \beta_2 = 0$ and $\alpha_3 = \beta_3 = -1$.

Next, α_4 and β_4 are roots of the quadratic equation

$$x^2 - x - 1 = 0$$

This gives

$$\alpha_4 = \frac{1+\sqrt{5}}{2}, \quad \beta_4 = \frac{1-\sqrt{5}}{2}$$

Similarly (and because $\psi_4 \neq \psi_5$),

$$\alpha_5 = \frac{1+\sqrt{5}}{2}, \quad \beta_5 = \frac{1-\sqrt{5}}{2}$$

This gives a complete table for A_5. Notice, that unlike in the case of S_5, the values of characters are not integers, in fact they are not even rational.

4. Induction and Frobenius reciprocity.

Let H be a subgroup of a finite group G.

Let U be a $\mathbb{C}[H]$-submodule of $\mathbb{C}[H] \subset \mathbb{C}[G]$. We let $U \uparrow G$ the $\mathbb{C}[G]$-submodule $\mathbb{C}[G]U$ of $\mathbb{C}[G]$

This $U \uparrow G$ is a $\mathbb{C}[G]$-submodule of $\mathbb{C}[G]$ called the $\mathbb{C}[G]$-submodule induced from U.

The following properties are left without proofs.

Proposition 4.1. (1) *If U and V are $\mathbb{C}[H]$-submodules of $\mathbb{C}[H]$ and U is $\mathbb{C}[H]$-isomorphic to V. Then $U \uparrow G$ is $\mathbb{C}[G]$-isomorphic to $V \uparrow G$.*

(2) *(this is the corrolary of the above) Let U be a $\mathbb{C}[H]$-submodule of $\mathbb{C}[H]$. Suppose that*

$$U = U_1 \oplus \cdots \oplus U_m$$

where U_is are $\mathbb{C}[H]$-submodules. Then

$$U \uparrow G = U_1 \uparrow G \oplus \cdots \oplus U_m \uparrow G$$

The second property allows to define the induced module for **any** $\mathbb{C}[H]$-module (it is always a direct sum of submodules of $\mathbb{C}[H]$).

Ona can define the induced representation by choosing the set of representatives for the coset space G/H, then form the direct sum

$$\oplus_{e \in G/H} eV$$

with a natural action of G.

If V is a $\mathbb{C}[H]$ module, then

$$\dim(V \uparrow G) = [G : H] \dim(V)$$

It is easy to see the following (which shows that the induction is transitive):

Theorem 4.2. *Suppose H and K are subgroups of G such that $H \subset K \subset G$. If U is a $\mathbb{C}[H]$-module, then*

$$(U \uparrow K) \uparrow G \cong U \uparrow G$$

A few examples of induced representations.

(1) Let 1_H be the trivial representation of H. Then $1_H \uparrow G$ is the premutation representation on G/H (the set of cosets.) That means, $1_H \uparrow G$ acts as $xH \mapsto gxH$.

For example the representation induced on S_n by the trivial representation of A_n is the 2-dimensional representation ρ as follows : Choose basis $\{e_{-1}, e_1\}$, then e_i is sent to $e_{\epsilon(\sigma)}$.

We see that $e_1 + e_{-1}$ is a stable subspace, so is $e_1 - e_{-1}$. Hence ρ is the sum of two one dimensional representations : the trivial one and the non-trivial one.

The induces representation induced by the trivial representation of the trivial subgroup

(2) The induced representation of the regular representation of H is the regular representation of G.

Another example of induced representation.

Take $G = S_3$. This is generated by $(1,2,3)$ and $\sigma = (1,2)$. Let H be the subgroup generated by $(1,2,3)$; it is cyclic of order three. Consider the usual representation $\rho_H\colon (1,2,3) \mapsto \zeta_3$ on the one-dimennsional vector space V. We know already that $\rho_H \uparrow G$ will be two dimensional.

We have

$$V \uparrow G = V \oplus \sigma V$$

Let v_1 be the basis of V and $v_2 = (1,2)v_1$.

Then 1 acts as the identity (this is always the case).

$$(1,2)v_1 = v_2, \quad (1,2)v_2 = v_1$$

Hence $(1,2)$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We have $(1,2,3)v_1 = \zeta_3 v_1$. Now, notice that $(1,2,3)(1,2) = (1,2)(1,2,3)^2 (= (1,3))$. This gives

$$(1,2,3)v_2 = (1,2,3)(1,2)v_1 = (1,2)(1,2,3)^2 v_1 = \zeta_3^2 (1,2)v_1 = \zeta_3^2 v_2$$

this gives

$$(1,2,3) \mapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$$

That completely determines the induced representation.

If ψ is a character of H, let U be an $\mathbb{C}[H]$-module of which ψ is a character. The the character $\psi \uparrow G$ of $U \uparrow G$ is called the induced character (from ψ).

Our aim is to prove the following theorem:

Theorem 4.3 (Frobenius reciprocity theorem)**.** *Let H be a subgroup of G. Let χ be a character of G and let ψ be a character of H. Then*

$$< \psi \uparrow G, \chi >_G = < \psi, \chi \downarrow H >_H$$

We will use a lemma:

Lemma 4.4. *Let V and W be $\mathbb{C}[G]$-modules with characters χ and ψ respectively. Then*

$$\dim(Hom_{\mathbb{C}[G]}(V, W)) = < \chi, \psi >$$

Proof. Write

$$V = V_1^{c_1} \oplus \cdots \oplus V_m^{c_m}$$

where V_is are simple submodules of $\mathbb{C}[G]$. Similarly

$$W = V_1^{d_1} \oplus \cdots \oplus V_k^{d_m}$$

By Shur's lemma

$$\dim(\mathrm{Hom}_{\mathbb{C}[G]}(V_i, V_j)) = \delta_{i,j}$$

It follows that

$$\dim(\mathrm{Hom}_{\mathbb{C}[G]}(V, W)) = \sum_{i=1}^{k} c_i d_i$$

But on the other hand:

$$\chi = \sum c_i \chi_i \text{ and } \psi = \sum d_i \chi_i$$

(where χ_is are all characters of G). It follows that

$$< \chi, \psi >= \sum c_i d_i$$

□

Another lemma.

Lemma 4.5. *Let U be a $\mathbb{C}[H]$-submodule of $\mathbb{C}[H]$ and V a $\mathbb{C}[G]$-submodule of $\mathbb{C}[G]$. Then*

$$\dim Hom_{\mathbb{C}[G]}(U \uparrow G, V) = \dim Hom_{\mathbb{C}[H]}(U, V \downarrow H)$$

Proof. Let $\phi \in \mathrm{Hom}_{\mathbb{C}[G]}(U \uparrow G, V)$ and let $\overline{\phi} \in \mathrm{Hom}_{\mathbb{C}[H]}(U, V \downarrow H)$ be the restriction of ϕ to U. The map

$$\phi \mapsto \overline{\phi}$$

is obviously $\mathbb{C}$-linear. **We will show that this is an isomorphism.**

We are going to use the following lemma:

Lemma 4.6. *Let G be a finite group and H a subgroup. Let U be a $\mathbb{C}[H]$-submodule of $\mathbb{C}[H]$. Let θ be a $\mathbb{C}[H]$-homomorphism from U to $\mathbb{C}[G]$. Then there exists an r in $\mathbb{C}[G]$*

$$\theta \colon u \mapsto ur$$

Proof. Let θ be a homomorphism $U \longrightarrow \mathbb{C}[G]$. Let W be a $\mathbb{C}[H]$-module such that $\mathbb{C}[G] = U \oplus W$. Define $\alpha \colon \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$ by $\alpha(u + w) = \theta(u)$.

Then α is an endomorphism of $\mathbb{C}[G]$, viewed as $\mathbb{C}[H]$-module.

Let $r = \alpha(1) \in \mathbb{C}[G]$. Then

$$\alpha(u) = \theta(u) = \alpha(1u) = \alpha(1)\alpha(u) = ru$$

because $u \in U \subset \mathbb{C}[H]$ □

First let us show the surjectivity.

Let $\phi \in \mathrm{Hom}_{\mathbb{C}[H]}(U, V \downarrow H)$. There exists $r \in \mathbb{C}[G]$ such that $\phi(u) = ur$. Define $\theta\colon U \uparrow G \longrightarrow \mathbb{C}[G]$ by

$$\theta(s) = sr$$

Then $\theta \in \mathrm{Hom}_{\mathbb{C}[G]}(U \uparrow G, V)$ and $\overline{\theta} = \phi$. This proves the surjectivity.

Now, suppose $ur_1 = ur_2$ for all $u \in U$, then $gur_1 = gur_2$ for all $g \in G, u \in U$, hence $sr_1 = sr_2$ for all $s \in U \uparrow G$. It follows that $\theta \longrightarrow \overline{\theta}$ is injective.

This finishes the proof. $\square$

Now, to derive the Frobenius reciprocity, we just write

$$\chi = \sum d_i \chi_i \text{ and } \psi = \sum e_j \psi_j$$

Then

$$< \psi \uparrow G, \chi >_G = \sum_{i,j} e_j d_i < \psi_j \uparrow G, \chi_i >_G = \sum_{i,j} e_j d_i < \psi_j, \chi_i \downarrow H >_H = < \psi, \chi \downarrow H >_H$$

This finishes the proof of Frobenius reciprocity.

Let us illustrate this with the example $G = S_3$.

We have three conjugacy classes and the character table is as follows:

g	1	$(1,2)$	(123)
$C_G(g)$	6	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

and the subgroup $H =< a = (123) >$ and let χ the character of $(123) \mapsto \zeta$. The conjugacy classes of H are of course $1, a, a^2$ and the values taken by χ are $1, \zeta, \zeta^2$. Let ψ_1, ψ_2, ψ_3 be the three irreducible characters of H. Here $\chi = \psi_2$.

We know that $\chi \uparrow G = \chi_3$. Its values are $2, 0, -1$.

Notice that because χ_3 is irreducible, $< \chi_3, \chi_2 >= \chi_3, \chi_1 >= 0$

The values taken by $\chi_3 \downarrow H$ are $2, -1, -1$. Notice that this is $\chi_3 \downarrow H = \psi_2 + \psi_3$.

We have

Let us calculate

$$< \chi \uparrow G, \chi_3 >=< \chi_3, \chi_3 >= 1$$

Now,

$$< \chi, \chi_3 \downarrow H >=< \psi_2, \psi_2 + \psi_3 >= 1 + 0 = 1$$

They agree as predicted by Frobenius reciprocity.

4.1. **Values of induced characters.** There is a simple way to evaluate the values of induced characters. Let ψ be a character of H and define the function $\psi\colon G \longrightarrow \mathbb{C}$ by $\psi(g)$ if $g \in H$ and 0 otherwise (we extends ϕ by zero.)

Proposition 4.7. *The values of $\psi \uparrow G$ are given by*

$$(\psi \uparrow G)(g) = \frac{1}{|H|} \sum_{y \in G} \psi(y^{-1}gy)$$

for $g \in G$.

Proof. Define $f(g) = \frac{1}{|H|} \sum_{y \in G} \psi(y^{-1}gy)$. We wish to prove that $f = \psi \uparrow G$. It is trivial to check that $f(w^{-1}gw) = f(g)$ hence f is a class function. Rememeber that irreducible characters form a basis of the vector space of class functions. To show that $f = \psi$, is suffices to check that

$$< f, \chi >_G = < \psi \uparrow G, \chi >_G$$

for all irreducible characters of G. Let χ be an irreducible character.

$$< f, \chi >_G = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi(g)} = \frac{1}{|G||H|} \sum_{g,y \in G} \psi(y^{-1}gy)\overline{\chi(g)}$$

Let $x = y^{-1}gy$. Then

$$< f, \chi >_G = \frac{1}{|G||H|} \sum_{x,y \in G} \psi(x)\overline{\chi(yxy^{-1})}$$

Now, $\psi(x) = 0$ if $x \notin H$ and $\chi(yxy^{-1}) = \chi(x)$ for all $y \in G$. Therefore

$$< f, \chi >_G = \frac{1}{|H|} \sum_{x \in H} \psi(x)\overline{\chi(x)} < \psi, \chi \downarrow H >_H$$

Now, by Frobenius reciprocity, $< f, \chi >_G = < \psi \uparrow G, \chi >_G$ which shows exactly that $f = \psi \uparrow G$. □

Corollary 4.8.

$$(\psi \uparrow G)(1) = \frac{|G|}{|H|}\psi(1)$$

This is immediate.

Let $x \in G$ Define a class function f_x^G on G by $f_x^G(y) = 1$ if $y \in x^G$ and 0 otherwise. (this is simply the characteristic function of the conjugacy class x^G).

Proposition 4.9. *Let χ be a character of G and $x \in G$. Then*

$$< \chi, f_x^G >_G = \frac{\chi(x)}{|C_G(x)}$$

Proof. This is an easy calculation.

$$< \chi, f_x^G >_G = \frac{1}{|G|} \sum_{g \in G} \chi(g) f_x^G(g) = \frac{1}{|G|} \sum_{g \in x^G} \chi(g)$$

$$= \frac{|x^G|}{|G|} \chi(x) = \frac{\chi(x)}{|C_G(x)|}$$

□

Proposition 4.10. *Let ψ be a character of $H \subset G$ and $x \in G$.*

(1) *if no element of x^G lies in H, then $(\psi \uparrow G)(x) = 0$*
(2) *if some element of x^G lies in H, then*

$$(\psi \uparrow G)(x) = |C_G(x)| \left(\frac{\psi(x_1)}{|C_G(x_1)|} + \cdots + \frac{\psi(x_m)}{|C_G(x_m)|} \right)$$

where $x_1, \ldots, x_m \in H$ and $f_x^G \downarrow H = f_{x_1}^H + \cdots + f_{x_m}^H$.

Proof. We have

$$\frac{(\psi \uparrow G)(x)}{|C_G(x)|} = < \psi \uparrow G, f_x^G >_G = < \psi, f_x^G \downarrow H >_H$$

If no element of x^G lies in H, then $f_x^G \downarrow H = 0$ and hence $(\psi \uparrow G)(x) = 0$

Otherwise

$$\frac{(\psi \uparrow G)(x)}{|C_G(x)|} = < \psi, f_{x_1}^H + \cdots + f_{x_m}^H >_H = |C_G(x)| \left(\frac{\psi(x_1)}{|C_G(x_1)|} + \cdots + \frac{\psi(x_m)}{|C_G(x_m)|} \right)$$

□