REPRESENTATION THEORY. EXAMPLES OF POSSIBLE EXAM QUESTIONS.

1

(1) State and prove the Shur's lemma for two simple R-modules.
(2) State Wedderburn's theorem and describe the steps in its proof.

2

(1) Let G be a finite group and F a field. Say what is meant by a representation of G over F. Let $G = S_3$. Construct a three-dimensional representation of G.
    Define the group algebra $F[G]$ (for G an abstract finite group). Is $F[G]$ a division algebra?
    Show that there is a one-to-one correspondence between representations of G over F and finitely generated $F[G]$-modules.
(2) State and prove Maschke's theorem.
    Suppose $F = \mathbb{C}$ and G is finite. Explain the decomposition of $\mathbb{C}[G]$ into simple submodules.
    What are the degrees of irreducible representations of the following groups: C_4, D_6, D_8, D_{10}.
    Write down explicitly all the irreducible representations of D_8.

3

Let G be a finite group V a $\mathbb{C}[G]$-module.

Say what is meant by the character χ of ρ.

Show that χ is independent of the choice of a basis for V.

Define the inner product of two class functions on G.

Let U and V be two *simple* $\mathbb{C}[G]$-modules with characters χ and ψ respectively. Show that $<\chi, \chi>= 1$ and $<\chi, \psi>= 0$.

Deduce that two $\mathbb{C}[G]$-modules are isomorphic if and only if they have the same character.

Write down row and column orthoganality relations.

Let G be a finite group that has four conjugacy classes and suppose we are given the following part of its character table:

g_i	g_1	g_2	g_3	g_4
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	α_1	α_2	α_3	α_4

where $\omega = e^{2i\pi/3}$.

Find the values α_i taken by χ_4.

4

Let V and W be two $\mathbb{C}[G]$-modules with characters χ_V and χ_W respectively. Define the tensor product $V \otimes W$ and show that the product $\chi_V \chi_W$ is a character.

Show that a product of a character by a linear character is irreducible.

The group S_5 has 7 conjugacy classes. Write down their representatives. Find two characters of degree one χ_1 and χ_2.

Calculate the premutation character χ_3 and show that it is irreducible.

Hence find another irreducible character χ_4.

5

(1) Let G be a finite group and H a subgroup. Let U be a $\mathbb{C}[G]$-module and V a $\mathbb{C}[H]$-module. Define the restriction $U \downarrow H$ and induction $V \uparrow G$. State the Frobenius reciprocity.
(2) Let $G = D_{10} = \{a, b : a^5 = b^2 = 1, b^{-1}ab = a\}$ and let H be the subgroup generated by a. Write down the character table of H. Calculate the induced characters, say which ones are irreducible.