SEMISIMPLE ALGEBRAS AND REPRESENTATION THEORY. EXERCISES 3.

ANDREI YAFAEV

(1) A group G of order 8 has the following character table.

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	?	?	?	?	?

Find the values of χ_5.

(2) G has four conjugacy classes and the following part of the character table.

g_i	$g_1 = 1$	g_2	g_3	g_4
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	α_1	α_2	α_3	α_4

Complete the table.

(3) Let χ_{reg} be the character of the regular representation of G.

Show that for any character χ

$$< \chi_{reg}, \chi >= \chi(1)$$

Let χ_1 be the trivial character.

Let χ be a *non-trivial* character such that for all g, $\chi(g)$ is *real* and $\chi(g) \geq 0$.

Show that $< \chi, \chi_1 >$ is *strictly positive* and conclude that χ is *reducible*.

(4) Let G and H be two finite groups with r and s conjugacy classes respectively. Let χ_is be irreducible characters of G and ψ_is those of H.

Show that $G \times H$ has exactly rs irreducible characters and they are $\chi_i\psi_j$.

(you need to show that the $\chi_i\psi_j$s are irreducible and that the number of conjugacy classes of $G \times H$ is rs)

Construct the character table of $S_3 \times C_2$.

(5) Let $G = S_4$ and let H be the subgroup $< (1,2,3,4),(1,3) >$. Show that H is isomorphic to D_8.

For each irreducible character of G, express $\chi \downarrow H$ as sum of irreducible characters.

(6) Suppose that G has an abelian subgroup H of index n. Show that for any irreducible character χ of G,

$$\chi(1) \leq n$$

(7) Let G be a finite group and let M be its character table viewed as an $r \times r$ matrix (r is the number of conjugacy classes).

Calculate $M\overline{M}^t$ and derive the expression for $|\det(M)|$.

(8) Let H be a subgroup of G and $\chi_1, \ldots, \chi_r$ irreducible characters of G.

Let ψ be an irreducible character of H and $d_1, \ldots, d_s$ integers such that $\psi \uparrow G = d_1\chi_1 + \cdots + d_k\chi_k$.

Show that

$$\sum_i d_i^2 \leq |G : H|$$

(9) Let $H \subset G$ be a subgroup. Show directly from the defintion that

$$(\psi \uparrow G)(1) = \frac{|G|}{|H|}\psi(1)$$